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Chapter 7.2.

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Probability background:

For a continuous random variable $X$, we call $F(x) = \Pr(X \leq x)$ the distribution function of $X$, and we call $f(x) = F'(x)$ the density function.

Note that $\Pr(a \leq X \leq b) = \int_a^b f(x) \, dx$.

The expected value is $E[X] = \int_a^b x f(x) \, dx$.
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End.
When we model random arrivals, and $X$ is the time between the random arrivals, a common probability model is the distribution

$$F(t) = \begin{cases} 
1 - e^{-\lambda t} & t \geq 0 \\
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\end{cases}$$

This distribution is called the exponential distribution with rate parameter $\lambda$. 

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In our setting (since $s, t > 0$) this translates to

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End
By the strong law of large numbers, the average of $n$ observations of $X_1, \ldots, X_n$ tends to $EX_n$ for large $n$. 

That is, $X_1 + X_2 + \cdots + X_n \rightarrow E(X_n)$ (1) as $n \rightarrow \infty$ with probability 1 (be careful: this is not quite point wise convergence; read about almost sure convergence).
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End
Try the Maple program below using various values of \( n \) for the Uniform distribution and the Normal distribution.
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```maple
restart;
with(Statistics):
X := RandomVariable(Uniform(0, 1));
X := RandomVariable(Normal(0, 1));
Mean(X);
Sample(X, 2);
Sample(X, 1);

k := 100;
x := 0;
for i from 1 to k do m := Sample(X, 1);
x := x + m[1];
do:
x/k;
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Example 7.3: A “type I counter” is used to measure radioactive decay. Decays occur at random, at an unknown rate, and the purpose of the counter is to determine the decay rate. Due to the mechanism in the counter, each decay observed locks the counter for a \( a = 3 \times 10^{-9} \) seconds, during which additional decays are not detected.
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Variables: $\lambda =$ decay rate (decays/second) $T_n =$ time of nth observed decay (seconds)
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Assumptions: Radioactive decays occur at random with rate $\lambda$. $T_{n+1} - T_n \geq 3 \times 10^{-9}$. Times between successive decays are independent and identically distributed: the distribution is assumed to be exponential with rate parameter $\lambda$. 

Objective: Find (or estimate) $\lambda$ on the basis of finitely many observations $T_1, \ldots, T_n$. 

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The random time $X_n$ consists of $a = 3 \times 10^{-9}$ seconds of lock time, and then a waiting period from then until the first decay after it comes unlocked.
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Let $Y_n$ denote this second part $X_n = a + Y_n$. 
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Because of the memoryless property, \( Y_n \) is also exponentially distributed with parameter \( \lambda \).
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To see this note that

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On the otherhand, if $X = a + Y$ then

$$Pr(X > t + a | X > a) = Pr(Y + a > t + a | Y + a > a) = Pr(Y > t)$$

thus $X_n$ and $Y_n$ have the same distribution.
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That is,

$$\frac{X_1 + X_2 + \cdots + X_n}{n} \to E(X_n) = E(a + Y_n) \quad (2)$$

$$= a + E(Y_n) \quad (3)$$

$$= a + 1/\lambda \quad (4)$$

as $n \to \infty$ with probability 1 (almost surely).
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\lambda = \frac{n}{(T_n - na)}.
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Conclusion:

To use the instrument to gauge a decay rate, we measure a large number of decays, take the total time until the nth observed decay $T_n$ and plug into the formula $\lambda = n\frac{T_n}{n-a}$. 

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