2.3 Linear 1st order

\[ a(x) y' + a_0(x) y = g(x) \]

Standard form:
(A) \[ y' + P(x) y = f(x) \]
claim: \( y \) is a soln of (A) if and only if it solves (B) and/or (C).

(B) \[ y' e^{ \int P(x) dx } + P(y)e^{ \int P(x) dx } = f(x)e^{ \int P(x) dx } \]

(C) \[ \frac{d}{dx}(y e^{ \int P(x) dx } ) = f(x)e^{ \int P(x) dx } \]
The term \( e^{\int P(x)\,dx} \) is called the integrating factor or I.F.

for \( y' + P(x)y = f(x) \)

Consider

(i) \( \frac{dP}{dt} = P(P-5) \)

(ii) \( \frac{dy}{dx} + 7xy = x^2 \)

(iii) \( \frac{dy}{dx} = 7xy \)

which can be solved by the I.F. method?

which can be solved by the separable variables method?
1st order DE's

- Solvable by Separable Variables
- Solvable by I.F. method
Solve

\[ xy' + 3x^3y' = x^3 \]

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**Soln**: The recommended process is to start with equation (C) on page 1.

**Process**

1. Put D.E. in Std. Form
2. Identify \( P(x) \) and determine \( e^{\int P(x)dx} \), the I.F.
3. Write eq'n (C)...
   \[
   \frac{d}{dx}(y e^{\int P(x)dx}) = f(x) e^{\int P(x)dx}
   \]
4. Solve the D.E. above
Solve \( xy' + 3x^3 y = x^3 \)

**Solution:**

1) **Standard Form** \( y' + 3x^2 y = x^2 \)

2) \( P(x) = 3x^2 \) so I.F. = \( e^{\int 3x^2 \, dx} \)

3) **Equation (c)**

\[
\frac{d}{dx}(ye^{x^3}) = x^2 e^{x^3}
\]

4) \( ye^{x^3} = \frac{1}{3} e^{x^3} + c \)

So \( y = \frac{1}{3} + ce^{-x^3} \)

The solution.
\[ \begin{cases} \frac{dy}{dx} + \frac{1}{x} y = e^x \\ y(1) = 2 \end{cases} \]  

1) \[ y' + \frac{1}{x} y = \frac{e^x}{x} \]  

2) \[ P(x) = \frac{1}{x} \text{ so I.F.} = e^{\int \frac{dx}{x}} = e^{\ln|x|} = |x| \]  

Because of the I.C. \( y(1) = 2 \)  

we use the interval \( (0, \infty) \)  

so I.F. = \( x \).  

3) \[ \frac{d}{dx}(xy) = xe^x = e^x \]  

4) \[ xy = e^x + c \]  

\[ y = \frac{e^x}{x} + \frac{c}{x} \]  

solution of D.E.
Continuation:

Now we solve the I. V. P. That is, we fit the I.C.

\[ y'(1) = 2 \]

The solution of the DE is

\[ y = \frac{e^x}{x} + \frac{c}{x} \]

\[ 2 = y(1) = \frac{e^1}{1} + \frac{c}{1} \quad \therefore c = 2 - e \]

So the solution of I.V.P. \[ y = \frac{e^x}{x} + \frac{2-e}{x} \]
2.3 General Terminology

Given

(D) \[ y' + P(x)y = f(x) \]

the new DE

(E) \[ y' + P(x)y = 0 \] is called

the associated homogeneous DE

Let's find the general sol'n of (D)

\[
\frac{d}{dx}(y e^{\int P(x) dx}) = f(x) e^{\int P(x) dx}
\]

\[ y = e^{-\int P(x) dx} \int f(x) e^{\int P(x) dx} \, dx + ce^{-\int P(x) dx} \]

is a one-parameter family of sol'n.s.

It is called the general sol'n of (D).
\[ y = e^{-Spdx} \int f(x) e^{Spdx} \, dx + ce^{-Spdx} \]

\[ y = y_c + y_p \quad \text{(general soln of (D))} \]

\[ y_c \text{ solve (E)} \]

\[ y_c \text{ is called the complementary soln} \]

\[ y_p \text{ is called the particular soln.} \]

\[ y_p \text{ solves (D) but is not the general soln.} \]
\[ y' + y = x \]

2) \( p(x) = 1 \), so I.F. = \( e^{\int p(x) \, dx} = e^x \)

3) \( \frac{dy}{dx}(y e^x) = x e^x \)

4) \( y e^x = \int x e^x \, dx \)  
\[ \text{Integration by parts.} \]

So \( y e^x = x e^x - e^x + c \)

\( y = (x e^x - e^x + c) e^{-x} \)

\( y = \frac{x - 1 + c e^{-x}}{y_e} \)