401 Linear Differential Equations
Through page 121.

I.V.P.
\[
\begin{align*}
& a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x) \\
& y(a) = y_a \\
& y'(a) = y'_a
\end{align*}
\]

B.V.P.
\[
\begin{align*}
& a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x) \\
& y(a) = y_a \\
& y(b) = y_b
\end{align*}
\]

What does the existence and uniqueness theorem say?
4.1 Terminology

"non-homogeneous"

\[ a_2(x) y'' + a_1(x) y' + a_0(x) y = g(x) \]

\[ \downarrow \text{ associated \ homogeneous \ DE} \]

\[ a_2(x) y'' + a_1(x) y' + a_0(x) y = 0 \]

**Superposition Principle (Theorem 4.1.2)**

If \( y_1 \) and \( y_2 \) are solutions of the homogeneous DE

\[ a_2(x) y'' + a_1(x) y' + a_0(x) y = 0 \]

then \( C_1 y_1 + C_2 y_2 \) is also a solution for any constants \( C_1 \) and \( C_2 \).
Differential operators

\[ D \leftrightarrow \frac{d}{dx} \quad D^2 \leftrightarrow \frac{d^2}{dx^2} \]

\[ D^0 \leftrightarrow \square \]

\[ a_n(x)y^{(n)} + a_1(x)y' + a_0(x)y \]
\[ = a_{2n}(x)D^2y + a_1(x)Dy + a_0(x)Iy \]
\[ = (a_2(x)D^2 + a_1(x)D + a_0(x)I) y \]
\[ L \]
\[ = L(y) . \]

example:
4.1.1

Almost definition 4.1.1

Two functions \( \{f_1(x), f_2(x)\} \)
are linearly dependent if there are constants \( c_1, c_2 \) not both zero such that
\[
c_1 f_1(x) + c_2 f_2(x) = 0.
\]

Otherwise, \( \{f_1(x), f_2(x)\} \) is linearly independent.
4.1

Wronskian

\[ W(f_1, f_2) = \begin{vmatrix} f_1 & f_2 \\ f'_1 & f'_2 \end{vmatrix} \]

\[ W(f_1, f_2, f_3) = \begin{vmatrix} f_1 & f_2 & f_3 \\ f'_1 & f'_2 & f'_3 \\ f''_1 & f''_2 & f''_3 \end{vmatrix} \]

**Theorem:** Let \( y_1, \ldots, y_n \) be solutions of an \( n \)-th order homogeneous linear DE. defined on an interval \( I \).

Then \( \{y_1, \ldots, y_2\} \) is linearly independent if and only if

\[ W(y_1, y_2, \ldots, y_n) \neq 0 \text{ on } I. \]
4.1 Fundamental Sets of Solutions

**Def 4.1.3** Any set of $n$ linearly independent solutions of

$$a_n(x)y^{(n)} + \ldots + a_1(x)y' + a_0(x)y = 0$$

is said to be a **fundamental set of solutions** (on some interval $I$)

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**General Solution:** If $\{y_1, \ldots, y_n\}$ is a fundamental set for $(*)$, then

$$y = c_1y_1 + c_2y_2 + \ldots + c_ny_n$$

is called the **general solution** of $(*)$

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**Theorem** Any solution of $(*)$ is equal to $(**)$ for some appropriate choice of $c_1, c_2, \ldots, c_n$.

*(Note: see hypotheses in textbook)*
4.1

e.g.: show that \( \{e^x, e^{3x}\} \) is a fundamental set of solutions for \( y'' + 4y' + 3y = 0 \).

Write the general soln of \( y'' + 4y' + 3y = 0 \).