4.3

- Linear
  - Constant coefficient
  - Homogeneous

In the language of Section 4.1, we now study how to construct fundamental solution sets for the type of DE stated.

In Section 4.4 we will consider non-homogeneous linear DE's. We will (at that time) need more terminology from 4.1.
Recall 1st order:
\[ ay'' + by' = 0 \]
\[ y' + \frac{b}{a}y = 0 \]

\[ \text{PCW constant} \]
\[ \text{Speedy} \]
\[ \frac{b}{a}x \]

I.F. = \( e^{\int \frac{b}{a}x} = e^{\frac{b}{a}x} \)
\[
\frac{d}{dx}(ye^{\frac{b}{a}x}) = 0 \cdot e^{\frac{b}{a}x} = 0
\]

\[ ye^{\frac{b}{a}x} = c \]
\[ y = ce^{-\frac{b}{a}x} \]

A soln:
\[ y' + ry = 0 \]

\[ y = e^{-rx} \]

(a fundamental set?)
Guess solutions of
\[ ay'' + by' + cy = 0 \]
have the form \( y = e^{mx} \)

\[ a[e^{mx}]'' + b[e^{mx}]' + c[e^{mx}] = 0 \]

\[ am^2e^{mx} + bem^{x} + ce^{mx} = 0 \]

\[ (am^2 + bm + c)e^{mx} = 0 \]

\[ [true \iff \text{only if}] \]

\[ am^2 + bm + c = 0 \]

Conclusion: \( y = e^{mx} \)

solves \( ay'' + by' + cy = 0 \)

\[ \iff \text{if and only if} \]

solves \( am^2 + bm + c = 0 \)

auxiliary equation
The DE
\[ ay'' + by' + cy = 0 \]
has auxiliary eqn \[ am^2 + bm + c = 0 \]
\[ y = e^{mx} \] solves \((*)\) if and only if \( m \) solves \((***)\).

**case 1:** \[ am^2 + bm + c = 0 \]
has distinct real roots \( m_1 \) and \( m_2 \).
then solutions are \( y_1 = e^{m_1x} \), \( y_2 = e^{m_2x} \)
General solution: \( y = c_1 e^{m_1x} + c_2 e^{m_2x} \)
Case 2: \( am^2 + bm + c = 0 \)
has repeated real root \( m_1 \).
\( y_1 = e^{m_1 x} \) to get a
2nd soln use the method of Section 4.02 (reduction of order method)
Case 3 \[ an^2 + bn + c = 0 \]
has complex roots (necessarily a conjugate pair)

\[ m_1 = \alpha + \beta i \quad m_2 = \alpha - \beta i \]
\[ e^{m_1 x} = e^{(a + bi)x} \]
\[ = e^{\alpha x} (\cos \beta x + i \sin \beta x) \]
\[ \frac{e^{m_2 x}}{e^{(\alpha - \beta) x}} = e^{\alpha x} (\cos \beta x - i \sin \beta x) \]
\[ e^{m_1 x}, e^{m_2 x} \text{ solns} \]
\[ \Rightarrow e^{m_1 x} + e^{m_2 x} \text{ is a soln} \]

\[ e^{m_1 x}, e^{m_2 x} \text{ solns} \]
\[ \Rightarrow e^{m_1 x} - e^{m_2 x} \text{ is a soln} \]
\[ \Rightarrow \]
Summary

\[ ay'' + by' + cy = 0 \]
\[ am^2 + bm + c = 0 \]

If the zeros of the auxiliary equation are:

1) real \( m_1 \neq m_2 \) then

\[ y = c_1 e^{m_1 x} + c_2 e^{m_2 x} \]

2) real repeated (zero) \( m \) then

\[ y = c_1 e^{mx} + c_2 xe^{mx} \]

3) complex \( \alpha + \beta i \) and \( \alpha - \beta i \)

then

\[ y = c_1 e^{\alpha x} \cos \beta x + c_2 e^{\alpha x} \sin \beta x \]

are the general solutions.