Section 4.6 Variation of Parameters

We assume a 2nd order linear differential equation

\[ a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x) \]  \hspace{1cm} (1)

has been written in the standard form

\[ y'' + P(x)y' + Q(x)y = f(x) \]

The Variation of Parameters formula is

\[ y_p(x) = y_1(x) \int \frac{-y_2(x)f(x)}{W(y_1(x), y_2(x))} \, dx + y_2(x) \int \frac{y_1(x)f(x)}{W(y_1(x), y_2(x))} \, dx \]

where \( y_1 \) and \( y_2 \) are a fundamental set of solutions of the associated homogeneous differential equation

\[ a_2(x)y'' + a_1(x)y' + a_0(x)y = 0. \]

The general solution of (1) has the form

\[ y = y_c + y_p = C_1 y_1 + C_2 y_2 + y_p \]
Example: Variation of Parameters

Find a particular solution of \( y'' + y = \tan(x) \)

Solution: \( y_p(x) = c_1 \cos(x) + c_2 \sin(x) \)

\[
W(y_1(x), y_2(x)) = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix} = 1
\]

\[
y_p(x) = -\cos(x) \int \sin(x) \tan(x) \, dx + \sin(x) \int \cos(x) \tan(x) \, dx
\]

\[
= -\cos(x) \int \frac{\sin^2(x)}{\cos(x)} \, dx + \sin(x) \int \sin(x) \, dx
\]

\[
= -\cos(x) \int \frac{1 - \cos^2(x)}{\cos(x)} \, dx - \sin(x) \cos(x)
\]

\[
= -\cos(x) \int \sec(x) \, dx + \cos(x) \int \cos(x) \, dx - \sin(x) \cos(x)
\]

\[
= -\cos(x) \sec(x) \, dx
\]

\[
= -\cos(x) \ln | \sec(x) + \tan(x) | + C
\]
A brief explanation of the method from the first page

Keeping the notation from the first page, we start by asking the question: are there functions \( u_1(x) \) and \( u_2(x) \) that such that

\[
y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)
\]

solves the ODE (3)?

\[
y'' + P(x)y' + Q(x)y = f(x)
\]

Substituting (2) into the ODE (3) one eventually obtains the conditions for success:

\[
y_1' u_1' + y_2' u_2' = 0
\]

\[
y_1' u_1' + y_2' u_2' = f(x)
\]

Matrix methods for solving systems of equations (Cramer’s rule for solving systems of equations) tell us that \( u_1' = W_1/W \) and \( u_2' = W_2/W \) where

\[
W_1 = \begin{vmatrix} 0 & y_2(x) \\ f(x) & y_2'(x) \end{vmatrix} = -y_2(x)f(x), \quad W_2 = \begin{vmatrix} y_1(x) & 0 \\ y_1'(x) & f(x) \end{vmatrix} = y_1(x)f(x)
\]

and where \( W \) is the wronskian of \( y_1 \) and \( y_2 \). It only remains to integrate \( u_1' \) and \( u_2' \).

\[
y_p = y_1 \int u_1' \, dx + y_2 \int u_2' \, dx
\]

which is the formula on the first page of these notes.

Note: this is called a variation of parameters method because we get the solution \( y_p \) by varying the \( C_1 \) and \( C_2 \) parameters in the general solution of the associated homogeneous problem \( y_c = C_1y_1 + C_2y_2 \)