Part 2

Free Damped Systems

\[ m \frac{d^2x}{dt^2} = -kx - \beta \frac{dx}{dt} \]

\[ m \frac{d^2x}{dt^2} + kx + \beta \frac{dx}{dt} = 0 \]

\[ \frac{d^2x}{dt^2} + \frac{\beta}{m} \frac{dx}{dt} + \frac{k}{m} x = 0 \]

\[ \frac{d^2x}{dt^2} + 2\alpha \frac{dx}{dt} + \omega^2 x = 0 \]
\[ x'' + \frac{B}{m} x' + \omega^2 x = 0 \]

Auxiliary eqn
\[ r^2 + \frac{B}{m} r + \frac{k}{m} = 0 \]

- or - \[ m r^2 + B r + k = 0 \]

\[ \text{Soln} \quad r = -\frac{B \pm \sqrt{B^2 - 4mk}}{2m} \]

\[ \text{Case 1:} \quad B^2 - 4mk > 0 \quad (\text{over damped}) \]

The general soln is ...

Physically, this means ...

\[ \text{Case 2:} \quad B^2 - 4mk = 0 \quad (\text{critically damped}) \]

The general soln is ...
Case 3: $\beta^2 - 4\mu k < 0$ (under damped)

\[
r = -\beta \pm \frac{\sqrt{\beta^2 - 4\mu k}}{2\mu}
\]

\[
= -\beta \pm \frac{\sqrt{4\mu k - \beta^2}}{2\mu}
\]

The general solution is --

Notice, we have sines and cosines.
This is the case where oscillation occurs!
6.1

Examples

Determine whether the system
is under-, over- or critically damped
by examining the auxiliary equation.

1) \( x'' + 8x' + 16x = 0 \)

Based on the roots of the
auxiliary e.g. \( r^2 + 8r + 16 = 0 \)...

2) \( x'' + 2x' + 10x = 0 \)

3) \( x'' + 6x' + 4x = 0 \)
ex: Suppose \( ax'' + bx' + cx = 0 \)

has auxiliary equation

\[(m - (-2 + 3i))(m - (-2 - 3i)) = 0\]

a). Then find the general soln

answer: \( x(t) = c_1 e^{2t} \cos(3t) + c_2 e^{2t} \sin(3t) \)

b) For \( c_1 = 3 \) and \( c_2 = 4 \),

put the soln into the form

\( x(t) = A \sin(\omega t + \phi) \)

c) discuss the maple plot on

the following page -
> C1 := 3;
> C2 := 4;
> A := sqrt(C1^2+C2^2);
> Phi := evalf(arctan(C1/C2));
> plot({A*exp(-2*t), -A*exp(-2*t), exp(-2*t)*A*sin(3*t+Phi)}, t=-Pi/4..Pi, thickness=3, view=-5..5);