8.2 \( p(\lambda) = (\lambda - \alpha I_1. \) Suppose \( p(\lambda) \)

has a repeated root.

**Case 1** \( \lambda_1 \leftrightarrow k_1. \) Suppose that

\[ \{k_1, k_2\} \]

is a set of two "distinct" eigenvectors for \( \lambda_1 \) — in other words — two linearly independent eigenvectors.

Then \( X_1 = e^{\lambda_1 t} k_1 \)

and \( X_2 = e^{\lambda_1 t} k_2 \)

yield two solutions that form a fundamental set \( \{X_1, X_2\} \)

and the general solution is

\[ X = c_1 X_1 + c_2 X_2 \]

**Case 2** \( \lambda \) only has one eigenvector. Trouble.
Recall section 4.3
\[ x'' - 6x' + 9x = 0 \]
\[ m^2 - 6m + 9 = (m-3)^2 = 0 \]
solutions: \[ x(t) = e^{3t} \]
and \[ x(t) = \square \]

Associated system:
Let \[ y = x' \]
\[
\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -9 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}
\]

Let's import solutions from the 2nd order eq'n above.
\[
X_1 = \begin{pmatrix} e^{3t} \\ (e^{3t})' \end{pmatrix} = \begin{pmatrix} e^{3t} \\ 3e^{3t} \end{pmatrix} = e^{3t} \begin{pmatrix} 1 \\ 3 \end{pmatrix} = e^{\lambda t} k_1,
\]

\[
X_2 = \begin{pmatrix} te^{3t} \\ (te^{3t})' \end{pmatrix} = \begin{pmatrix} te^{3t} \\ e^{3t} + 3te^{3t} \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} te^{3t} + e^{3t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = k_1 + a \text{ new vector } p
\]

\[
= k_1 te^{\lambda t} + e^{\lambda t} p
\]
(2x2 systems)

**Conclusions:** If \( \lambda_1 \) only has one "distinct" eigenvector, then solutions are

\[
X_1 = e^{\lambda_1 t} k_1,
\]

\[
X_2 = t e^{\lambda_1 t} k_1 + e^{\lambda_1 t} p.
\]

How do we determine \( P \)?

\( P \) solves:

\[
(A - \lambda_1 I) P = k_1,
\]

A special vector to be determined below.
Example:
Consider $X' = \begin{pmatrix} 3 & -18 \\ 2 & -9 \end{pmatrix} X$.

$p(\lambda) = (\lambda + 3)^2$ so $\lambda = -3$ is a repeated eigenvalue.

The eigenvector for $\lambda = -3$ is $(3, 1)$ (or any nonzero multiple).

$X_1 = e^{-3t} \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ is one solution.

A second (linearly independent) solution is

$X_2 = t e^{-3t} \begin{pmatrix} 3 \\ 1 \end{pmatrix} + e^{-3t} P$

where $P$ solves

$(A - (-3)I)P = k_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$
\[(A - (-3)I)P = K,\]
\[
\begin{bmatrix}
3 & -18 \\
2 & -9
\end{bmatrix}
- 
\begin{bmatrix}
-3 & 0 \\
0 & -3
\end{bmatrix}

\begin{bmatrix}
P_1 \\
P_2
\end{bmatrix}
=
\begin{bmatrix}
3 \\
1
\end{bmatrix}
\]
\[
\begin{bmatrix}
6 & -18 \\
2 & -6
\end{bmatrix}

\begin{bmatrix}
P_1 \\
P_2
\end{bmatrix}
=
\begin{bmatrix}
3 \\
1
\end{bmatrix}
\]
\[
\begin{bmatrix}
6 & -18 & | & 3 \\
2 & -6 & | & 1
\end{bmatrix}
\sim
\begin{bmatrix}
6 & -18 & | & 3 \\
0 & 0 & | & 0
\end{bmatrix}
\]
\[
\sim
\begin{bmatrix}
1 & -3 & | & \frac{1}{2} \\
0 & 0 & | & 0
\end{bmatrix}
\]
\[
P_1 - 3P_2 = \frac{1}{2}
\]
\[
P_1 = \frac{1}{2} + 3P_2
\]
\[
P = \begin{bmatrix}
\frac{1}{2} + 3P_2 \\
P_2
\end{bmatrix}
= \begin{bmatrix}
\frac{1}{2} \\
0
\end{bmatrix}
+ P_2 \begin{bmatrix}
3 \\
1
\end{bmatrix}
\]
\[
= \begin{bmatrix}
\frac{1}{2} \\
0
\end{bmatrix}
+ P_2K,
\]
we let \(P_2 = 0\)
for convenience.