Smooth Nielsen periodic point theory

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Nielsen Theory and Related Topics 2009
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Let $M$ be a compact, connected manifold, $f : M \to M$ continuous.

The classical problem in the fixed point theory is to find minimal number of fixed points in the homotopy class of $f$. *Nielsen number* $N(f)$, by the definition, is a topological invariant that gives the lower bound for the number of fixed points in the homotopy class of $f$:

$$\forall g \sim f \quad \#\text{Fix}(g) \geq N(f).$$
The question, posed by Nielsen, is whether this is the best lower bound, i.e.

\[ \exists g \sim f \quad \# \text{Fix}(g) = N(f) ? \]

was answered positively by Wecken in 1942, for manifolds with dimension at least 3.
Estimation of the number of periodic points

The problem may be formulated more generally: could one defined a topological invariant which allows to estimate from below the number periodic points.

Let $r \in \mathbb{N}$ be fixed, such an invariant should be less or equal

$$\min_{g \sim f} \#\text{Fix}(g^r).$$
Nielsen type number of period $r$

In 1983 Boju Jiang defined *Nielsen type number of period $r$*, $NF_r[f]$, and showed that this is homotopy invariant and lower bound for the number of $r$-periodic points in homotopy class of $f$:

$$\forall g \sim f \quad \#\text{Fix}(g^r) \geq NF_r[f].$$

J. Jezierski in 2000-3 r. proved that if $\dim M \geq 3$, then this is the best lower bound:

$$\exists g \sim f \quad \#\text{Fix}(g^r) = NF_r[f],$$

which means that $NF_r[f]$ is the minimal number of $r$-periodic points in the homotopy class of $f$:

$$NF_r[f] = \min_{g \sim f} \#\text{Fix}(g^r).$$
The aim of smooth Nielsen periodic point theory is to define and calculate a topological invariant that would be a counterpart of $NF_r[f]$ in smooth category $(C^1)$, i.e. the invariant of smooth homotopy: $\sim$:

$$\min_{g \sim f} \# \text{Fix}(g^r) = ?$$
It turned out that this problem is strictly related to the question what are possible forms of local indices (at an isolated fixed point) of iterations of maps homotopic to $f$. 
Fixed point index

- $\text{ind}(f, x_0) \in \mathbb{Z}$ - topological invariant used to detect fixed points - geometrically it describes how many times $id - f$ "winds" the neighborhood of 0 around this point - locally $\text{deg}(id - f, 0)$.
- In the problem of minimizations of fixed points the crucial role plays the sequence of indices of iterations

$$\text{ind}\{ (f^n, x_0) \}_{n=1}^{\infty},$$

(provided it is well-defined).
Can \( \text{ind}\{(f^n, x_0)\}_{n=1}^{\infty} \) take any integer values?

In 1971 Krasnosel’skii and Zabreiko noticed that (for continuous maps) for any prime number \( p \) holds:

\[
\text{ind}(f, x_0) \equiv \text{ind}(f^p, x_0) \pmod{p}.
\]

In 1984 A. Dold found much more general congruences for indices, called Dold relations:

\[
\sum_{k \mid n} \mu(n/k)\text{ind}(f^k, x_0) \equiv 0 \pmod{n},
\]

where \( \mu \) is the Möbius function.
Definicja

For a given $k \in \mathbb{N}$ we define

\[
\text{reg}_k(n) = \begin{cases} 
  k & \text{if } k \mid n, \\
  0 & \text{if } k \nmid n.
\end{cases}
\]

In other words, \(\text{reg}_k\) is the periodic sequence:

\[(0, \ldots, 0, k, 0, \ldots, 0, k, \ldots),\]

where the non-zero entries appear for indices divisible by \(k\).
Periodic expansion

A sequence of indices of iterations has the so-called periodic expansion,

$$\text{ind}(f^n, x_0) = \sum_{k=1}^{\infty} a_k \text{reg}_k(n),$$

where

$$a_n = \frac{1}{n} \sum_{k|n} \mu(n/k) \text{ind}(f^k, x_0),$$

$\mu$ is the Möbius function.

As a result, by Dold relations,

$$a_n \in \mathbb{Z}.$$
Restrictions for indices of continuous maps

Are there any further restrictions, except for Dold relations, for local indices of a continuous map $f : \mathbb{R}^m \to \mathbb{R}^m$?

In dimension $m = 1$ there are just few sequences of indices of a very special form.

For $m \geq 2$ there are no further restrictions, i.e. every sequence which satisfies Dold relations can be realized as a sequence of indices of iterations of some continuous map (Babenko, Bogatyi: $m \geq 3$, GG & P. Nowak-Przygodzki 2003: $m = 2$).
Chow, Mallet-Paret and Yorke showed in 1981 that there are very strong restrictions for \( \{\text{ind}(f^n, 0)\}_{n=1}^\infty \), if \( f \) is a \( C^1 \) map, and conjectured that there are no more restrictions.

GG and P. Nowak-Przygodzki (2006) confirmed the conjecture and listed all sequences of integers that could be indices of iterations of a smooth map in \( \mathbb{R}^3 \).
Let $M$ be a compact, smooth manifold of dimension $\geq 3$.

Boju Jiang in 1981 [Fixed point classes from a differential viewpoint] proved that Nielsen fixed point theory for smooth and continuous maps coincide.

This means that if $f$ is a smooth map, then it can always be smoothly deformed to a map $g$ with only $N(f)$ fixed points.

But it turns out that the smooth periodic points theory is quite different from the continuous one and the strong differences hold even in the simply-connected case.
Simply-connected continuous case

Assume additionally that $M$ is simply-connected. First study classical (continuous) category.

**Theorem**

Let $r$ be fixed natural number. Any continuous self-map $f$ of $M$ is homotopic to a map $g$ such that

$$\text{Fix}(g^r) = \begin{cases} \emptyset & \text{if } L(f^n) = 0 \text{ for all } n \mid r, \\ \{*\} & \text{otherwise}, \end{cases}$$

where $\{*\} = \text{Fix}(g)$ denotes the set which consists of one point.
Example

The antipodal map of $S^3$ is homotopic to a map which has only one fixed point.
Reducing $\text{Fix}(f^r)$ to a single point.
Assume that a self-map $f$ of $M$ is homotopic to a smooth map $g$ with $\text{Fix}(g^r) = \{\ast\}$, then for every $n|r$ we have:

$$L(f^n) = L(g^n) = \text{ind}(g^n) = \text{ind}(g^n, \ast).$$

As a consequence, the necessary condition under which $f$ is smoothly homotopic to a map with one fixed point is that the sequence of Lefschetz numbers $\{(L(f^n))_{n|r}\}$ must be locally realized as a sequence of fixed point indices at an isolated fixed point $\ast$.

But these sequences have very special form! [by Chow, Mallet-Paret, York theorem], so it could take place only in some special cases, unlike the continuous case.
Sufficient condition

The necessary condition turns out to be the sufficient condition, so if

\[ \{(L(f^n))_{n|r} = \{\text{ind}(h^n, x_0)\}_{n|r} \]

for some \( x_0 \) and smooth \( h \),

then \( f \) is smoothly homotopic to a map with one fixed point.
The idea of the proof

**Creating Procedure** enables one to create an additional orbit in the homotopy class of $f$, by a homotopy $f_t$ which is constant near periodic points of $f$ (up to the given period $r$) and such that $f_1$ near the created orbit may be given by an arbitrarily prescribed formula.

**Canceling Procedure** enables one to remove in the homotopy class subsets of periodic points which have indices of iterations equal to zero.
Application of the procedures

Using Creating Procedure we create a fixed point $x_0$ such that
$$\{(L(f^n))_{\, n \mid r} = \{\text{ind}(h^n, x_0)\}_{\, n \mid r}.\]

Now $f_1$ has more periodic points: except of the old ones $\text{Old}$ there is $\{x_0\}$ and the ones which were added during the deformation $\text{Def}$.

As $\{(\text{ind}(h^n, x_0))_{\, n \mid r}$ realized Lefschetz numbers, the indices of $\text{Old} \cup \text{Def}$ are equal to zero and so may be removed by Canceling Procedure.
What is happening if the condition is not satisfied?

Then we seek for the representation of \( \{(L(f^n))_n\mid r \} \) as the minimal sum of sequences of indices of iterations realized on fixed points or periodic orbits.
We decompose

\[
\{(L(f^n))_n \mid r = c_1(n) + \ldots + c_s(n),
\]

where \( c_i = \text{ind}(h_i^n, P_i) \), \( P_i \) denotes some \( l_i \)-orbit.

Each such decomposition determines the number \( l = l_1 + \ldots + l_s \).
We define the number \( D_r^m[f] \), *periodic Dold number of the order* \( r \),
as the smallest \( l \) which can be obtained in this way.
Theorem

\( D^m_r[f] \) is the invariant we looked for:

\[
\min_{g \sim f} \# \text{Fix}(g^r) = D^m_r[f].
\]

In other words \( D^m_r[f] \) is the best lower bound for \( \# \text{Fix}(g^r) \) for \( g \) in the smooth homotopy class of \( f \).
Three things needed to compute the invariant

\[ \{ (L(f^n))_{n|r} = c_1(n) + \ldots + c_s(n), \]

each \( c_i(n) \) has the form \( \sum_{k=1}^{\infty} a_k \text{reg}_k(n) \).

- [1] We should know what is the exact form of \( c_i(n) \), i.e. know the list of sequences of local indices.
- [2] We have to know the right hand-side of the above formula, i.e periodic expansion of Lefschetz numbers.
- [3] Finally, we should solve the combinatorial problem: what is the minimal number of sequences \( c_i(n) \) that gives \( \{ (L(f^n))_{n|r} \).
Indices of iterations in dimension 3

Theorem

Item [1], known in $\mathbb{R}^3$, below the list of all sequences of indices in dimension 3:

(A) $c_A(n) = a_1\text{reg}_1(n) + a_2\text{reg}_2(n)$,
(B) $c_B(n) = \text{reg}_1(n) + a_d\text{reg}_d(n)$,
(C) $c_C(n) = -\text{reg}_1(n) + a_d\text{reg}_d(n)$,
(D) $c_D(n) = a_d\text{reg}_d(n)$,
(E) $c_E(n) = \text{reg}_1(n) - \text{reg}_2(n) + a_d\text{reg}_d(n)$,
(F) $c_F(n) = \text{reg}_1(n) + a_d\text{reg}_d(n) + a_{2d}\text{reg}_{2d}(n)$, where $d$ is odd,
(G) $c_G(n) = \text{reg}_1(n) - \text{reg}_2(n) + a_d\text{reg}_d(n) + a_{2d}\text{reg}_{2d}(n)$, where $d$ is odd.
Computing of $D^3_r[f]$

Item [2], periodic expansion for Lefschetz numbers is known for some type of 3-manifolds. For these manifolds we was able to execute the task described in item [3], so find $D^3_r[f]$. $D^3_r[f]$ was computed for:

- $S^2 \times I$,
- $S^3$,
- two-holed 3-dimensional closed ball.
Invariant of the space

$D_r^3[f]$ is often almost independent of $f$, it is insensitive to the homotopy class of $f$.

For example, if $r$ is odd and $f$ is a self-map of $S^3$ with $|\text{deg}(f)| > 1$, then $D_r^3[f] \in \{\zeta(r) - 1, \zeta(r)\}$, where $\zeta(r)$ is the number of divisors of $r$.

This follows from the simply-connectedness of $S^3$ and fast grow of Lefschetz numbers of iterations.

As a consequence, in some cases $D_r^3[f]$ may be perceived as an invariant of the whole space rather than of the homotopy class of $f$. 
Orbits of Reidemeister classes $\text{OR}(f^k)$

For each pair of numbers $l|k$ we define $i_{k,l} : \text{OR}(f^l) \to \text{OR}(f^k)$.

If $N^l \subset \text{Fix}(f^l)$, $N^k \subset \text{Fix}(f^k)$ are Nielsen classes representing Reidemeister classes $A^l \subset \text{OR}(f^l)$ and $A^k \subset \text{OR}(f^k)$ respectively, then $N^l \subset N^k$ implies $i_{k,l}(A^l) = A^k$. 
Orbits of Reidemeister classes $\mathcal{OR}(f^k)$

**Definition**

We say that for two orbits of Reidemeister classes $A \in \mathcal{OR}(f^k)$ and $B \in \mathcal{OR}(f^l)$, $B$ is preceding $A$ if $l|k$ and $i_{k,l}(B) = A$.

We write then $B \preceq A$ ($B \prec A$ if $B \preceq A$ but $A \neq B$).

The depth of an orbit $A$ is the smallest $l$ such that $A$ belongs to the image of $i_{k,l}$. 
Definition

For $B \in OR(f^k)$ we define the function $Reg_B : OR_\infty(f) \rightarrow \mathbb{Z}$ putting

$$Reg_B(A) = \begin{cases} 
k & \text{for } B \leq A, \\
0 & \text{otherwise.} \end{cases}$$
Generalized periodic expansion

There exist unique numbers $a_B \in \mathbb{Z}$ such that

$$\text{ind}(f^n; A) = \sum_B a_B \cdot \text{Reg}_B(A)$$

for all $A \in \mathcal{OR}(f^n)$.

We may create a periodic orbit in each orbit of Reidemeister classes $H$, which gives the sequence of indices $c$ of a given admissible pattern (which comes from locally smooth map).
The sequence $c$ gives impact to Reidemister orbits

$$C_H(A) = \begin{cases} \ c(n) & \text{for } H \preceq A; A \in OR(f^n), \\ \ 0 & \text{otherwise}. \end{cases}$$  \hspace{1cm} (1)$$

We say then that the sequence $c$ is *attached* at the orbit $H \in OR(f^h)$. 
Invariant $NJ D^m_r[f]$ 

\[
\text{ind}(f^n; A) = \sum_B a_B \text{Reg}_B = C_{H_1} + \cdots + C_{H_s},
\]

where $C_{H_i}$ corresponds to the sequence $c_i$ attached at the class $H_i \in OR(f^{h_i})$. 

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Smooth Nielsen periodic point theory
Invariant $\text{NJD}^m_r[f]$

Definition

We define $\text{NJD}^m_r[f]$, the Nielsen-Jiang-Dold number, by:

$$\text{NJD}^m_r[f] = \text{minimal sum } h_1 + \cdots + h_s,$$

such that the above equality holds.
Consider a directed graph in which vertices are orbits of Reidemeister classes and a (unique) directed edge from $B$ to $A$ corresponds to the relation $B \prec A$.

If we associate with each vertex $A \in OR(f^k)$ the number $a_A$ from the generalized periodic expansion then we get, Reidemeister graph $GOR(f)$. 
For a fixed integer $r$ we denote by $GOR(f; r)$ the full subgraph whose vertices are elements of $OR(f^k)$ for $k|r$.

$GOR(f; r)$ carries all data needed to determine $NJD_r[f]$. 

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Smooth Nielsen periodic point theory
$NF_r(f)$ versus $NJD_r[f]$

If $f$ is a continuous map, then the minimal number of points in $\text{Fix}(g')$ for all $g$ homotopic to $f$ is equal to $NF_r(f)$ - the classical invariant introduced by Jiang.

What is the difference between $NF_r(f)$ and $NJD_r[f]$?

It could be explained by a use of Reidemeister graph.
First, we briefly remind the definition of $NF_r(f)$. We call a subset $S \subset OR_r(f)$ *Preceding System* if each essential orbit in $OR_r(f)$ is preceded by an orbit in $S$. 
Definition

$S$ is called *Minimal Preceding System* (MPS) if the sum of the depth of elements in $S$

$$\sum_{H \in S} d(H)$$

is minimal.

The number $NF_r(f)$ is defined as the above least sum i.e. the sum of depth of orbits in an MPS.
Now, notice that for calculating $NF_r(f)$ we do not care about the values of indices at vertices of the graph, the only information we need is whether the indices are non-zero (the class is essential) or not.

Calculating $NJD_r[f]$ we have to realize also indices in each vertex $B$, which are expressed by the coefficients $a_B$ at $\text{Reg}_B$. 
If, during the calculation of $\text{NJD}_r[f]$, we attach in each $H$ of a given MPS some admissible sequence, that may be not enough, because some coefficients $a_B$ at $\text{Reg}_B$ may be not realized.

As a consequence, usually $\text{NJD}_r[f] > NF_r(f)$ and the equality holds only in very special situations.
Example

Let \( f : \mathbb{RP}^3 \to \mathbb{RP}^3 \) be a map of degree \( d = 3 \).

Let us fix \( r = 6 \). Then \( OR(f^n) = \mathbb{Z}_2 \), and we can draw the following Reidemeister graph:
Real Projective space
Real Projective space

It follows from the graph that the (unique) MPS is \( \{1', 1'', 2''\} \), hence

\[
NF_6(f) = 1 + 1 + 2 = 4.
\]

On the other hand,

\[
NJD_6[f] = 7.
\]
### Summary

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Publications


Computer programs

The definitions of $D_r^m[f]$ and $NJD_r^m[f]$ have operational character and in many cases may be reduced to combinatorial procedure.

As a result, very promising would be create algorithms which allow to device a computer program to calculate the invariants.
Our team (GG, J. Jezierski and P. Nowak-Przygodzki) is very close to give the description of indices of iterations for smooth self-maps of $\mathbb{R}^m$ for any $m$.

This would allow us to calculate $D_r^m[f]$ and $NJD_r^m[f]$ for self-maps of manifolds of arbitrary dimension.
For two-dimensional manifolds, in general even $NF_r[f]$ is not the best lower bound for the number of $r$-periodic points.

Nevertheless, it would be interesting to investigate what is happening in the smooth category in dimension 2.