A remark on fixed points of graph maps

Boju Jiang
(joint with Shida Wang and Qiang Zhang)

International Conference on Nielsen Theory and Related Topics
St. John’s, Newfoundland, Canada
June 9–13, 2009
Consider self-maps of a graph (= finite 1-dimensional cell complex). Bestvina and Handel in their paper “Train tracks and automorphisms of free groups” (Annals of Math. 1992) established bounds on the ‘rank’ of fixed point classes. Later, Jiang (Math. Annalen 1997) gave bounds on the index of fixed point classes. We now give new bounds which unify and sharpen these results.
Outline

1. Nielsen fixed point class
   - Fixed point class
   - Index and stabilizer
   - Invariance

2. Previous bounds
   - Characteristic
   - Bounds on characteristic
   - Bounds on index

3. New bounds
   - Mixed bounds
   - Comparison
   - Types of fixed point classes

4. Surface maps
   - Bounds for aspheric surfaces
   - Surface automorphisms

Boju Jiang (joint with Shida Wang and Qiang Zhang)
Let $X$ be a connected compact polyhedron, and $f : X \to X$ a self-map.

Definition (for nonempty fixed point class)

Two fixed points $x, x' \in \text{Fix } f$ are in the same fixed point class if and only if there is a path $c$ (called a Nielsen path) from $x$ to $x'$ such that $c \simeq f \circ c$ rel endpoints.
Let $p : \tilde{X} \to \tilde{X}$ be the universal covering of $X$, with group $\pi$ of covering translations.

**Definition**

For any lifting $\tilde{f} : \tilde{X} \to \tilde{X}$ of $f : X \to X$, the projection of its fixed point set is called a **fixed point class** of $f$, written $F = p(\text{Fix} \tilde{f})$. Two fixed point classes $p(\text{Fix} \tilde{f})$ and $p(\text{Fix} \tilde{f}')$ are the same $\iff$ there exists $\gamma \in \pi$ such that $\tilde{f}' = \gamma^{-1} \circ \tilde{f} \circ \gamma$.

Note that a fixed point class is always labeled by a conjugacy class of liftings. Two **empty fixed point classes** may have different labels even if they are set-wise equal.
The fixed point set $\text{Fix } f$ decomposes into a disjoint union of fixed point classes.

**Definition**

The **index** of a fixed point class $F$ is defined by the fixed point index

$$\text{ind}(F) := \text{ind}(f, F) := \sum_{x \in F} \text{ind}(f, x).$$

The summation is meant for $F$ consisting of isolated fixed points. Empty fixed point classes have $\text{ind} = 0$. 

Boju Jiang (joint with Shida Wang and Qiang Zhang )

Fixed points of graph maps
Stabilizer of a fixed point class

Definition (for nonempty fixed point class)

For a fixed point \( x \in F \), consider the stabilizer subgroup

\[
\text{Stab}(F) := \{ \gamma \in \pi_1(X, x) \mid \gamma = f_\pi(\gamma) \},
\]

where \( f_\pi : \pi_1(X, x) \to \pi_1(X, x) \) is the induced endomorphism. It is independent of the choice of \( x \in F \), up to isomorphism.
Stabilizer of a fixed point class

Each lifting $\tilde{f}$ induces an endomorphism $\tilde{f}_\pi : \pi \to \pi$ defined by

$$\tilde{f} \circ \gamma = \tilde{f}_\pi(\gamma) \circ \tilde{f}.$$ 

Definition

For a fixed point class $F = \rho(\text{Fix } \tilde{f})$, its stabilizer subgroup is

$$\text{Stab}(F) := \{ \gamma \in \pi \mid \gamma = \tilde{f}_\pi(\gamma) \} = \{ \gamma \in \pi \mid \tilde{f} \circ \gamma = \gamma \circ \tilde{f} \}.$$ 

Up to group isomorphism, it is independent of the choice of $\tilde{f}$.

This definition works for empty fixed point classes as well.
A homotopy $H = \{h_t\} : f_0 \simeq f_1 : X \to X$ gives rise to a natural one-one correspondence

$$H : F_0 \leftrightarrow F_1$$

from the fixed point classes of $f_0$ to the fixed point classes of $f_1$.

Remark. A homotopy may create or remove fixed point classes. The correspondence is one-one only when empty fixed point classes are taken into account.

**Theorem (HOMOTOPY INVARIANCE)**

*Under the correspondence via a homotopy $H$,*

$$\text{ind}(f_0, F_0) = \text{ind}(f_1, F_1) \quad \text{and} \quad \text{Stab}(f_0, F_0) \cong \text{Stab}(f_1, F_1).$$
Suppose \( \phi : X \to Y \) and \( \psi : Y \to X \) are maps. Then \( \psi \circ \phi : X \to X \) and \( \phi \circ \psi : Y \to Y \) are said to differ by a \textit{commutation}. The map \( \phi \) sets up a natural one-one correspondence

\[
F_X \leftrightarrow F_Y
\]

from the fixed point classes of \( \psi \circ \phi \) to the fixed point classes of \( \phi \circ \psi \).

\textbf{Theorem (COMMUTATION INVARIANCE)}

Under the correspondence via commutation,

\[
\text{ind}(\psi \circ \phi, F_X) = \text{ind}(\phi \circ \psi, F_Y), \quad \text{Stab}(\psi \circ \phi, F_X) \cong \text{Stab}(\phi \circ \psi, F_Y).
\]
Suppose $G$ is a group of finite type, i.e. there is a finite cell complex $Y$ such that $\pi_i(Y) = 0$ for all $i \neq 1$ and $\pi_1(Y) \cong G$. Since all such $Y$ have the same homotopy type, we define the characteristic of $G$ to be the Euler characteristic of $Y$.

Notation: $\text{chr}(G) := \chi(Y)$.

**Example**

- trivial group: $\text{chr}(\{1\}) = 1$.
- infinite cyclic group: $\text{chr}(\mathbb{Z}) = 0$.
- free group of rank $n$: $\text{chr}(F_n) = 1 - n$.
- surface group $\pi_1(S)$: $\text{chr}(\pi_1(S)) = \chi(S)$. 
From now on, unless otherwise stated, we always assume $X$ to be a connected finite graph, and $f : X \rightarrow X$ is a selfmap.

**Definition**

The rank and characteristic of a fixed point class $F$ is defined as

\[
\text{rank}(F) := \text{rank}(\text{Stab}(F)), \\
\text{chr}(F) := \text{chr}(\text{Stab}(F)).
\]

**Example**

For the identity map $\text{id}_X : X \rightarrow X$, the whole $X$ is a fixed point class. Now $\text{chr}(X)$ equals to the Euler characteristic $\chi(X)$ of $X$. All other (empty) fixed point classes have characteristic 0.
Theorem A

For fixed point classes of a graph selfmap, we have:

1. If $\text{chr}(F) < 0$, then $F$ is nonempty.
2. $\sum_{\text{chr}(F) < 0} \text{chr}(F) \geq \chi(X)$.

Theorem A was first proved for $f$ a homotopy equivalence. The “Scott Conjecture” $\text{chr}(F) \geq \chi(X)$ follows as a corollary.

M. Bestvina, M. Handel, Train tracks and automorphisms of free groups.
Theorem B

For a graph selfmap \( f : X \to X \), we have:

1. \( \text{ind}(F) \leq 1 \) for every fixed point class \( F \) of \( f \).
2. \( \sum_{\text{ind}(F) < -1} \{\text{ind}(F) + 1\} \geq 2\chi(X) \).

It started as an observation in Thurston’s theory of surface automorphisms. Its proof for graph maps uses Bestvina-Handel. Theorem B also holds true for surface maps.

We get a new result.

**Theorem C**

*For a graph selfmap $f : X \to X$, we have:*

1. $\text{ind}(F) \leq \text{chr}(F)$ for every fixed point class $F$ of $f$.
2. $\sum_{\text{ind}(F) + \text{chr}(F) < 0} \{\text{ind}(F) + \text{chr}(F)\} \geq 2\chi(X)$.

The proof resembles that of Theorem B. First reduce the Theorem to the case of injective $f_{\pi}$, via commutations. Then use Bestvina-Handel’s theory of train track maps which also works for monomorphisms of free groups.
Comparison

Theorem C implies both Theorems A and B. Compare to see the improvement.

(C1): \( \text{ind}(F) \leq \text{chr}(F) \).

(A1): If \( \text{chr}(F) < 0 \), then \( F \) is nonempty.

(B1): \( \text{ind}(F) \leq 1 \).

(C2): \[ \sum_{\text{ind}(F) + \text{chr}(F) < 0} \{ \text{ind}(F) + \text{chr}(F) \} \geq 2\chi(X). \]

(A2): \[ \sum_{2\,\text{chr}(F) < 0} 2\,\text{chr}(F) \geq 2\chi(X). \]

(B2): \[ \sum_{\text{ind}(F) + 1 < 0} \{ \text{ind}(F) + 1 \} \geq 2\chi(X). \]
### Types of fixed point classes

<table>
<thead>
<tr>
<th></th>
<th>ind</th>
<th>1</th>
<th>0</th>
<th>−1</th>
<th>−2</th>
<th>…</th>
<th>χ</th>
<th>…</th>
<th>2χ</th>
<th>2χ−1</th>
</tr>
</thead>
<tbody>
<tr>
<td>chr</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>■</td>
<td>○</td>
<td>■</td>
<td>□</td>
<td>□</td>
<td>□</td>
<td>□</td>
<td>□</td>
<td>□</td>
<td>□</td>
</tr>
<tr>
<td>0</td>
<td>□</td>
<td>○</td>
<td>□</td>
<td>□</td>
<td>□</td>
<td>□</td>
<td>□</td>
<td>□</td>
<td>□</td>
<td>□</td>
</tr>
<tr>
<td>−1</td>
<td>□</td>
<td>□</td>
<td>□</td>
<td>□</td>
<td>□</td>
<td>□</td>
<td>□</td>
<td>□</td>
<td>□</td>
<td>□</td>
</tr>
<tr>
<td>−2</td>
<td>□</td>
<td>□</td>
<td>□</td>
<td>□</td>
<td>□</td>
<td>□</td>
<td>□</td>
<td>□</td>
<td>□</td>
<td>□</td>
</tr>
<tr>
<td>2χ</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2χ−1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Table: Types of fixed point classes**

Boju Jiang (joint with Shida Wang and Qiang Zhang)
Let $R_n$ be the rose with $n$ petals, with one vertex $\ast$ and $n$ edges. The group $\pi_1(R_n, \ast)$ has a free basis represented by the petals, denoted $\{a_1, a_2, \ldots, a_n\}$.

**Example**

Let $f_k : (R_n, \ast) \to (R_n, \ast)$, $0 \leq k \leq n$, be the map such that

$$f_{k\pi} : \begin{cases} a_i \mapsto a_i & \text{if } i \leq k, \\ a_i \mapsto a_i^2 & \text{if } i > k. \end{cases}$$

Then the fixed point class $\{\ast\}$ has rank(\ast) = $k$, chr(\ast) = $1 - k$, and ind(\ast) = $1 - 2n + k$. Thus $\{\ast\}$ lies on the lower right edge of the triangular region.
Bounds for aspheric surfaces

Aspheric surfaces are surfaces with infinite fundamental groups.

**Theorem D**

*For a selfmap $f: X \to X$ of a compact aspheric surface, we have:*

1. $\text{ind}(F) \leq \text{chr}(F)$ for every fixed point class $F$ of $f$.
2. $\sum_{\text{ind}(F) + \text{chr}(F) < 0} \{\text{ind}(F) + \text{chr}(F)\} \geq 2\chi(X)$.

If $f$ is homotopic to a diffeomorphism, then Thurston’s canonical form can be used to verify the bounds. If not, $f$ can be modified by homotopy and commutation to a graph map, then use Theorem C.
For a surface automorphism $f$, we may assume it is in Thurston canonical form.

A fixed point class $F$ with nontrivial $\text{Stab}(F)$ can come either

1. from an identity piece (periodic piece of period 1), or
2. from a reducing curve.

So $\text{Stab}(F)$ measures the potential shape of $F$. 
For a surface automorphism $f$, the inequality (C2) becomes an equality if and only if in the Thurston canonical form of $f$, each periodic piece is of period 1, and each pseudo-Anosov piece keeps all prongs into themselves. Thus every surface automorphism has an iterate for which (C2) is an equality.