The R-infinity property for infra-nilmanifolds

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The Reidemeister number and Almost–crystallographic groups

Pairs of maps between almost–crystallographic groups

2 and 3–dimensional crystallographic groups
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The R-infinity property for infra-nilmanifolds

Topological background

∀a ∈ A(\tilde{X}, p_X), \tilde{f} \circ a is another lift of f
∀b ∈ A(\tilde{Y}, p_Y), b \circ \tilde{f} is another lift of f
We obtain a morphism of groups

f_x : A(\tilde{X}, p_X) → A(\tilde{Y}, p_Y) : a ↦ f_x(a) defined by \tilde{f} \circ a = f_x(a) \circ \tilde{f}
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Topological background

\[
\begin{array}{ccc}
\tilde{X} & \overset{\tilde{f}}{\longrightarrow} & \tilde{Y} \\
p_X \downarrow & & \downarrow p_Y \\
X & \overset{f}{\longrightarrow} & Y
\end{array}
\]

\[
\forall a \in A(\tilde{X}, p_X), \quad \tilde{f} \circ a \text{ is another lift of } f
\]

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\forall b \in A(\tilde{Y}, p_Y), \quad b \circ \tilde{f} \text{ is another lift of } f
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\[ \begin{array}{ccc}
\tilde{X} & \xrightarrow{\tilde{f}} & \tilde{Y} \\
\downarrow{p_X} & & \downarrow{p_Y} \\
X & \xrightarrow{f} & Y
\end{array} \]

\[ \forall a \in A(\tilde{X}, p_X), \tilde{f} \circ a \text{ is another lift of } f \]
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\[ f_\times: A(\tilde{X}, p_X) \rightarrow A(\tilde{Y}, p_Y) : a \mapsto f_\times(a) \text{ defined by } \tilde{f} \circ a = f_\times(a) \circ \tilde{f} \]
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\[ \forall a \in A(\tilde{X}, p_X), \tilde{f} \circ a \text{ is another lift of } f \]

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We obtain a morphism of groups

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Let $f, g : X \rightarrow Y$ be maps.

$$\text{Coin}(f, g) = \{ x \in X \mid f(x) = g(x) \}.$$ 

For lifts $\tilde{f}, \tilde{g} : \tilde{X} \rightarrow \tilde{Y}$, we have that

$$p_X(\text{Coin}(\tilde{f}, \tilde{g})) \subseteq \text{Coin}(f, g)$$

and

$$\text{Coin}(f, g) = \bigcup_{\tilde{f}, \tilde{g}} p_X(\text{Coin}(\tilde{f}, \tilde{g})) = \bigcup \tilde{f} p_X(\text{Coin}(\tilde{f}, \tilde{g})).$$

$\Rightarrow$ division of coincidence set into classes.

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The R-infinity property for infra-nilmanifolds
Coincidence sets

Let $f, g : X \rightarrow Y$ be maps.

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$\Rightarrow$ division of coincidence set into classes.
Let $\varphi, \psi : G_1 \to G_2$ be two morphisms. Determine equivalence relation on $G_2$:

$$g \sim g' \iff \exists h \in G_1 : g = \psi(h)g'\varphi(h)^{-1}$$

Equivalence classes are “Reidemeister classes”: $R[\varphi, \psi]$. Reidemeister number $R(\varphi, \psi) = \#R[\varphi, \psi]$.

Take $\varphi : G \to G$, then $R(\varphi) = R(\varphi, \text{Id})$.

$G$ has the $R_\infty$ property if and only if $R(\varphi) = \infty$ for all automorphisms $\varphi$. 
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The R-infinity property for infra-nilmanifolds
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Equivalence classes are “Reidemeister classes”: $\mathcal{R}[\varphi, \psi]$. Reidemeister number $R(\varphi, \psi) = \#\mathcal{R}[\varphi, \psi]$.

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The $R$-infinity property for infra-nilmanifolds
Back to topology

\[ f, g : X \to Y \rightsquigarrow \tilde{f}, \tilde{g} : \tilde{X} \to \tilde{Y} \rightsquigarrow f_x, g_x : A(\tilde{X}, p_X) \to A(\tilde{Y}, p_Y) \]

Definition

\[ R(f, g) = R(f_x, g_x). \]

Let \( \gamma \in A(\tilde{Y}, p_Y) \), then \( p_X(\text{Coin}(\gamma \circ \tilde{f}, \tilde{g})) \) is stable on the whole Reidemeister class of \( \gamma \).

Definition

\( p_X(\text{Coin}(\gamma \circ \tilde{f}, \tilde{g})) \) is coincidence class induced by the Reidemeister class \( [\gamma] \).

Two different Reidemeister classes cannot induce the same non-empty coincidence class.
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Back to topology

$f, g : X \rightarrow Y \rightsquigarrow \tilde{f}, \tilde{g} : \tilde{X} \rightarrow \tilde{Y} \rightsquigarrow f_\times, g_\times : A(\tilde{X}, p_X) \rightarrow A(\tilde{Y}, p_Y)$

**Definition**

$$R(f, g) = R(f_\times, g_\times).$$

Let $\gamma \in A(\tilde{Y}, p_Y)$, then $p_X(\text{Coin}(\gamma \circ \tilde{f}, \tilde{g}))$ is stable on the whole Reidemeister class of $\gamma$.

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Two different Reidemeister classes cannot induce the same non-empty coincidence class.
Crystallographic groups

Isom(ℝⁿ) = ℝⁿ × O(n)

**Definition**

A uniform and discrete subgroup $E \subseteq ℝⁿ × O(n)$ is a crystallographic group.

$E$ torsionfree $\leadsto$ Bieberbach group $\leadsto$ Flat manifold $E \backslash ℝⁿ$.

**Example:**

$E = \left\langle \left[ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \mathbb{I}_2 \right], \left[ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \mathbb{I}_2 \right], \left[ \begin{pmatrix} 0 & 1/2 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \right] \right\rangle$
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First Bieberbach Theorem

Theorem (First Bieberbach Theorem)

\[ E \text{ crystallographic} \Rightarrow E \cap \mathbb{R}^n \cong \mathbb{Z}^n \text{ is a lattice of } \mathbb{R}^n \text{ and } \frac{E}{(E \cap \mathbb{R}^n)} = \frac{E}{\mathbb{Z}^n} \text{ is finite.} \]

Short exact sequence:

\[ 1 \to \mathbb{Z}^n \to E \to F \to 1. \]

\( F \) = holonomy group

Remark: \( F \cong \{ A \in O(n) \mid \exists a \in \mathbb{R}^n : (a, A) \in E \} \).

In previous example: \( F = \left\{ I_2, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \right\} \cong \mathbb{Z}_2 \).
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Second Bieberbach Theorem

Theorem (Second Bieberbach Theorem)

Let $E_1, E_2 \subseteq \text{Isom}(\mathbb{R}^n)$ be crystallographic. If $\varphi : E_1 \rightarrow E_2$ is an isomorphism, then $\exists (d, D) \in \text{Aff}(\mathbb{R}^n) = \mathbb{R}^n \rtimes \text{GL}(n, \mathbb{R})$ s.t.

$$\forall \alpha \in E_1 : \varphi(\alpha) = (d, D)\alpha(d, D)^{-1}.$$

Generalization

Theorem (K.B. Lee '95)

Let $E_1 \subseteq \text{Isom}(\mathbb{R}^n)$ and $E_2 \subseteq \text{Isom}(\mathbb{R}^m)$ be crystallographic. If $\varphi : E_1 \rightarrow E_2$ is a morphism, then $\exists d \in \mathbb{R}^m$ and a linear map $D : \mathbb{R}^n \rightarrow \mathbb{R}^m$ s.t. $\forall \alpha \in E_1 : \varphi(\alpha) \circ (d, D) = (d, D) \circ \alpha$.

Here $(d, D) : \mathbb{R}^n \rightarrow \mathbb{R}^m : x \mapsto d + Dx.$
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Here $(d, D) : \mathbb{R}^n \to \mathbb{R}^m : x \mapsto d + Dx.$
Replace $\mathbb{R}^n$ by $G$, be a connected and simply connected nilpotent Lie group.

Replace $\text{Aff}(\mathbb{R}^n) = \mathbb{R}^n \rtimes \text{GL}(n, \mathbb{R})$ by $G \rtimes \text{Aut}(G)$.

Action of $G \rtimes \text{Aut}(G)$ on $G$: $(a, A) g = a \cdot A(g)$

Replace $O(n)$ by any (maximal) compact subgroup $C \subseteq \text{Aut}(G)$.

Hence $\text{Isom}(\mathbb{R}^n) = \mathbb{R}^n \rtimes O(n) \cong G \rtimes C$. 
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From flat manifolds to infra-nilmanifolds

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Almost–Crystallographic groups

Definition

A uniform and discrete subgroup $E \subseteq G \rtimes C$ is an almost–crystallographic group (AC–group).

$E$ torsionfree $\leadsto$ almost–Bieberbach group $\leadsto$ infra-nilmanifold $E \backslash G$.

If $E \subseteq G$ ($E$ is a uniform lattice of $G$), then $E \backslash G$ is a nilmanifold.
First Generalized Bieberbach Theorem

**Theorem (L. Auslander ’60)**

\[ E \text{ almost-crystallographic} \Rightarrow N = E \cap G \text{ is a uniform lattice of } G \text{ and } E/(E \cap G) = E/N \text{ is finite.} \]

Short exact sequence (again \( F = \text{holonomy group} \)):

\[ 1 \to N \to E \to F \to 1. \]

Remark: \( F \cong \{ A \in C \subseteq \text{Aut}(G) \mid \exists a \in G : (a, A) \in E \} \).

Let \( \mathfrak{g} \) be Lie algebra of \( G \):

\[ \text{Aut}(G) \cong \text{Aut}(\mathfrak{g}) \hookrightarrow \text{Aut}(\mathfrak{g}) \subseteq GL(\mathbb{R}^n). \]

In this way we obtain the holonomy representation:

\[ \rho : F \to \text{Aut}(\mathfrak{g}) \subseteq GL(\mathbb{R}^n) : A \mapsto A_. \]
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Theorem (K.B.Lee ’95)

Let $E_1 \subseteq G_1 \rtimes \text{Aut}(G_1)$ and $E_2 \subseteq G_2 \rtimes \text{Aut}(G_2)$ be AC–groups. If $\varphi : E_1 \rightarrow E_2$ is a morphism, then $\exists d \in G$ and a continuous morphism $D : G_1 \rightarrow G_2$ s.t. $\forall \alpha \in E_1 : \varphi(\alpha)(d, D) = (d, D)\alpha$.

Where $(d, D) : G_1 \rightarrow G_2 : x \mapsto d \cdot D(x)$.

Corollary

Any continuous map $f : E_1 \backslash G_1 \rightarrow E_2 \backslash G_2$ between two infra-nilmanifolds is homotopic to a map induced by an affine map $(d, D) : G_1 \rightarrow G_2$. 
Theorem (K.B.Lee ’95)

Let $E_1 \subseteq G_1 \rtimes \text{Aut}(G_1)$ and $E_2 \subseteq G_2 \rtimes \text{Aut}(G_2)$ be AC–groups. If $\varphi : E_1 \to E_2$ is a morphism, then $\exists d \in G$ and a continuous morphism $D : G_1 \to G_2$ s.t. $\forall \alpha \in E_1 : \varphi(\alpha)(d, D) = (d, D)\alpha$.

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Corollary

Any continuous map $f : E_1 \backslash G_1 \to E_2 \backslash G_2$ between two infra–nilmanifolds is homotopic to a map induced by an affine map $(d, D) : G_1 \to G_2$. 
Corollary

Let \( f : E_1 \backslash G_1 \to E_2 \backslash G_2 \) be a continuous map between two infra-nilmanifolds and let \((d, D) : G_1 \to G_2\) be a homotopy lift of \( f \). Denote by

\[
\rho : F_1 \to \text{Aut}(\mathfrak{g}_1) \subseteq \text{GL}(\mathbb{R}^n) \quad \text{and} \\
\rho' : F_2 \to \text{Aut}(\mathfrak{g}_2) \subseteq \text{GL}(\mathbb{R}^m)
\]

the associated holonomy representations. Then there exists a map \( \phi : F_1 \to F_2 \) such that

\[
\rho'(\phi(A))D_* = D_*\rho(A) \quad \text{for all } A \in F_1
\]
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3. 2 and 3-dimensional crystallographic groups

Karel Dekimpe: The R-infinity property for infra-nilmanifolds
Step 1: A technical lemma

Lemma

Suppose we have a commutative diagram

\[ \begin{array}{ccccccccc}
1 & \rightarrow & H_1 & \rightarrow & G_1 & \rightarrow & F_1 & \rightarrow & 1 \\
\varphi' & \downarrow & \psi' & \rightarrow & \varphi & \downarrow & \psi & \rightarrow & 1 \\
1 & \rightarrow & H_2 & \rightarrow & G_2 & \rightarrow & F_2 & \rightarrow & 1 \\
\end{array} \]

where \( F_1 \) and \( F_2 \) are finite groups.

Then \( R(\varphi, \psi) < \infty \) if and only if \( R(\mu_\alpha \circ \varphi', \psi') < \infty, \forall \alpha \in G_2 \).

Here \( \mu_\alpha : H_2 \rightarrow H_2 : h \mapsto \alpha h \alpha^{-1} \).
Step 2: The Reidemeister number for morphisms of Lie groups

Let $G_1, G_2$ be simply connected, connected nilpotent Lie groups and $\mathfrak{g}_1, \mathfrak{g}_2$ be the associated Lie algebras.

Let $\Psi, \Phi : G_1 \to G_2$ be morphisms of Lie groups and $\Psi^*, \Phi^* : \mathfrak{g}_1 \to \mathfrak{g}_2$ be the associated Lie algebra morphisms.

**Observation:** If $\Psi^* - \Phi^* : \mathfrak{g}_1 \to \mathfrak{g}_2$ is surjective, then $\Psi \cdot \Phi^{-1} : G_1 \to G_2 : x \mapsto \Psi(x)\Phi(x)^{-1}$ is surjective.

In this case $R(\Phi, \Psi) = 1$. 
Step 2: The Reidemeister number for morphisms of Lie groups

$G_1, G_2$ simply connected, connected nilpotent Lie groups
$\mathfrak{g}_1, \mathfrak{g}_2$ associated Lie algebras.

Let $\psi, \phi : G_1 \to G_2$ be morphisms of Lie groups and $\psi_* : \mathfrak{g}_1 \to \mathfrak{g}_2$ be the associated Lie algebra morphisms.

Observation: If $\psi_* - \phi_* : \mathfrak{g}_1 \to \mathfrak{g}_2$ is surjective, then $\psi \cdot \phi^{-1} : G_1 \to G_2 : x \mapsto \psi(x)\phi(x)^{-1}$ is surjective.

In this case $R(\phi, \psi) = 1$. 
Conversely

**Lemma**

\[ R(\Phi, \Psi) = \infty \]

\[ \psi^* \circ \phi^* : g_1 \rightarrow g_2 \text{ is not surjective} \]
Let $\psi, \Phi : G_1 \to G_2$ be morphisms of Lie groups and $\psi_*, \Phi_* : \mathfrak{g}_1 \to \mathfrak{g}_2$ be the associated Lie algebra morphisms.

Let $\Lambda_1 \subseteq G_1$ (resp. $\Lambda_2 \subseteq G_2$) be a uniform lattice of $G_1$ (resp. $G_2$), with $\Phi(\Lambda_1) \subseteq \Lambda_2$ and $\Psi(\Lambda_1) \subseteq \Lambda_2$.

**Lemma**

Let $\varphi = \Phi|_{\Lambda_1}$ and $\psi = \Psi|_{\Lambda_1}$, then

$$R(\varphi, \psi) = \infty \iff \Psi_* - \Phi_* \text{ is not surjective}.$$
Step 3: The Reidemeister number for morphisms of lattices Lie groups

$G_1$, $G_2$ simply connected, connected nilpotent Lie groups $\mathfrak{g}_1$, $\mathfrak{g}_2$ associated Lie algebras.

Let $\psi, \Phi : G_1 \to G_2$ be morphisms of Lie groups and $\psi_*, \Phi_* : \mathfrak{g}_1 \to \mathfrak{g}_2$ be the associated Lie algebra morphisms.

Let $\Lambda_1 \subseteq G_1$ (resp. $\Lambda_2 \subseteq G_2$) be a uniform lattice of $G_1$ (resp. $G_2$), with $\Phi(\Lambda_1) \subseteq \Lambda_2$ and $\psi(\Lambda_1) \subseteq \Lambda_2$.

Lemma

Let $\varphi = \Phi|_{\Lambda_1}$ and $\psi = \psi|_{\Lambda_1}$, then

$$R(\varphi, \psi) = \infty \iff \psi_* - \Phi_* \text{ is not surjective.}$$
Step 4: The Reidemeister number for morphisms of almost–crystallographic groups

$E_1 \subseteq G_1 \rtimes \text{Aut}(G_1)$ and $E_2 \subseteq G_2 \rtimes \text{Aut}(G_2)$

$\varphi, \psi : E_1 \to E_2$ morphisms, induced by $(d_\varphi, D_\varphi), (d_\psi, D_\psi) : G_1 \to G_2$.

**Theorem**

$$R(\varphi, \psi) = \infty \iff \exists A \in F_2 : D_\psi^* - A^* D_\varphi^* \text{ is not surjective}$$
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General picture

\[ E \subseteq G \rtimes \text{Aut}(G), \, \varphi : G \to G, \text{ induced by } (d, D) : G \to G \]

Corollary

\[ R(\varphi) = \infty \iff \exists A \in F : \det(I - A_\ast D_\ast) = 0. \]

Corollary

\[ E \text{ (or } E \backslash G\text{) has the } R_\infty \text{ property if and only if } \forall \varphi : G \to G \text{ automorphism (induced by } (d, D)) \text{ there exists } A \in F \text{ with } \det(I - A_\ast D_\ast) = 0. \]
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General picture

$E \subseteq G \rtimes \text{Aut}(G), \varphi : G \to G$, induced by $(d, D) : G \to G$

Corollary

$R(\varphi) = \infty \iff \exists A \in F : \det(\mathbb{I} - A_\ast D_\ast) = 0$.

Corollary

$E$ (or $E \setminus G$) has the $R_\infty$ property if and only if $\forall \varphi : G \to G$ automorphism (induced by $(d, D)$) there exists $A \in F$ with $\det(\mathbb{I} - A_\ast D_\ast) = 0$. 

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The R-infinity property for infra-nilmanifolds
Does a crystallographic group $E$ have the $R_\infty$ property?

- Choose $E \subset \text{Isom}(\mathbb{R}^n) \subseteq \text{Aff}(\mathbb{R}^n)$ crystallographic.
- Determine the group $F \subseteq \text{GL}(n, \mathbb{R})$. (Here $A = A_*$)
- Determine all possible $D \in \text{GL}(n, \mathbb{R})$ such that there is a $d \in \mathbb{R}^n$ with $(d, D) : \mathbb{R}^n \rightarrow \mathbb{R}^n$ inducing a morphism on $E$.
- I.e. for some map $\varphi : F \rightarrow F$:
  $$\rho(\varphi(A))D_* = D_*\rho(A) \rightsquigarrow \phi(A)D = DA.$$  
- Check whether or not there is an $A \in F$ with $\det(I - AD) = 0$. 

Karel Dekimpe
Theorem

In dimension 2 there are 3 crystallographic groups (out of 17) not having the $R_{\infty}$ property.

(This result was also obtained by D.L. Gonçalves and P. Wong)

Theorem

In dimension 3 there are 12 crystallographic groups (out of 219) not having the $R_{\infty}$ property.
Theorem

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THANK YOU