Multiaxial Fatigue Analysis in a High-speed Microgenerator Rotor Shaft

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Abstract: Fatigue evaluation is one of the most concerning issue in the design of mechanical parts subject to alternative forces. Fatigue life estimation is more complicated when loads act on different axes. This paper shows the stress analysis result of the steel shaft of a high-speed microgenerator rotor with 47750 rpm and 100kW which has been carried out with analytical stress formulation. The shaft is subjected to alternative torsional and bending stresses. First, Mean and alternative components of stresses have been calculated; then, using a MATLAB code, multiaxial fatigue life has been calculated with equivalent static yield criteria and Sines method. Having considered five potential critical points on the shaft, fatigue life calculation is done in each critical point. Using different criteria, it is possible to compare results and decide about the fatigue strength of different sections. Consideration of the mean and alternative stresses effects with different axes is the major attempt in this paper.

Key words: Multiaxial fatigue; Mean stress; Alternative stress; Fatigue life; Life prediction

1 Introduction

In the design of rotating machinery, rotor fatigue analysis is one of the most important measures to be taken; this is because rotor is often subjected to various static and dynamic loads that may cause rotor to fail. If loads act on different axis, multiaxial fatigue analysis is necessary, because the effect of non-proportionality and mean stresses together can be considered. When the loads are in elastic limit, high cycle fatigue analysis should be accounted for accordingly. First attempts, to understand the phenomena of combined bending and torsion loading, was made by Gough [1]. Then Sines [2] introduced a criterion to measure the effect of mean stress in combined loading. One of drawbacks of the above criterion is that the effect of phase difference on high cycle fatigue life can not be recorded. The Criteria suggested by Findley [3] and McDiarmid [4] on the bases of critical plane can consider the effect of phase difference and mean stress together. In the special subject of shaft fatigue, References [5, 6 and 7] discuss various methods for predicting fatigue life of a round steel shaft subjected to in-phase torsion-bending loads. A multiaxial damage parameter based on maximum shear strain is proposed in the Ref. [8] which could predict life of notched shafts under bending and torsion. References [9, 10 and 11] discuss the strain based multiaxial fatigue analysis of SAE 1045 notched steel shaft using different critical plane approaches. Fatigue strength of notched steel shaft subjected to in-phase and out-of-phase tension and torsion loading, using an energy-based approach, is discussed in Ref. [12]. Previous attempts mainly focus on the strain or energy based life estimation approaches; but the main focus in this paper is on stress based life analysis with consideration of mean centrifugal stresses. Another difference is high rotational speed of shaft, which will create high frequency of stress alternation.

2 Rotor geometry and applied forces

Rotor is composed of central shaft, magnets, spacers and carbon fibre composite containment layer. Magnets are fixed on shaft body by composite layer. The shaft has a total Length of 25 cm and maximum diameter of 8 cm. Fatigue analysis is carried out on critical points, which have stress concentration and are potentially suspicious to be the location of fatigue failure. Figure 1 shows the isometric view of shaft with dimensions and Figure 2 shows the proposed critical points on the shaft.
Three kinds of forces are applied to the shaft: 

1) Alternating torsion
2) Fully reversed bending by transverse forces
3) Centrifugal forces

These forces and different kind(s) of resulting stresses are separately analyzed in the following section. All stresses components are calculated using a MATLAB code.

3 Stress analysis

Input torque and resistance of magnetic field, induce alternating shear stresses in the shaft. Mean torque distribution is shown in Figure 3.

Because the right end of the shaft is free, no torsional load is applied to this side; therefore critical points are analyzed on left and middle of the shaft. According to the results of magnetic field calculation [13], amplitude imposed torque is considered as 25 percent of mean input torque.

Bending stresses are created by rotor weight, structural unbalance and magnetic unbalance. According to the standard, this kind of rotor is categorized in grade 2.5 of structural unbalance. Resulting structural unbalance force is calculated in the following form [14]:

\[ F_{\text{unbalance( structural)}} = U_{\text{per}} \delta \omega^2 = 1000 Gr m \omega \]  

In which \( U_{\text{per}} \) is permissible unbalance, \( Gr \) is structural unbalance grade number, \( m \) is shaft mass and \( \omega \) is rotational speed.

Transverse magnetic forces are calculated as:

\[ F_{\text{unbalance(magnetic)}} = K_0 (e_0 + y) \]  

in which \( e_0 \) is misalignment and \( K_0 \) is calculated from Equation (3),

\[ K_0 = 3 \times 10^4 \left( \frac{D I_\delta}{\delta} \right) \]  

in which \( D \) is maximum diameter of rotor, \( I_\delta \) is equivalent rotor length and \( \delta \) is air gap length between rotor and stator; \( y \) is rotor deflection by magnetic force and rotor weight which is calculated from Equation (4),

\[ y = \frac{k_0 e_0 + W}{K - K_0} \]  

in which \( W \) is rotor weight and \( K \) is structural stiffness of rotor. In the worst situation if three mentioned forces locate in same direction, bending stress are calculated in the following form:

\[ \sigma_\text{z} = K_m \left( \frac{F_{\text{total}} x r}{2I} \right) \]  

in which:

\( (K_m) \) is stress intensity factor, \( r \) is rotor radius, \( l \) is rotor length and \( x \) is distance from support which here is the bearing. Figure 5(a) shows the fully reversed bending stress due to bending and Figure 5(b) shows alternating torsional stresses.

Magnetic unbalance is created due to misalignment of rotor and stator axis. This misalignment impose unsymmetrical magnetic field (B) and unsymmetrical magnetic pull (\( T_n \)) which are shown in Figure 4.
\[ \sigma_r = \frac{3 + \nu}{8} (b^2 - r^2) \rho \omega^2 \]
\[ \sigma_\theta = \frac{3 + \nu}{8} (b^2 + 3r^2) \rho \omega^2 \]
\[ (\sigma_r)_{m} = -\nu(\sigma_r + \sigma_\theta) \]

For calculating stresses around holes which create high stress concentration, stress relationship of a plate with central hole and biaxial tension is calculated from Equation (7):
\[ (\sigma_\rho)_{m} = \sigma_{\rho}(1 - 2\cos(2\theta)) + \sigma_\rho(1 - 2\cos(2\theta - \pi)) \]

in which \( \sigma_r \) and \( \sigma_\theta \) are calculated from Equations 4 and 5 respectively at the point with the radius equal to radius of the central point of the hole.

Assuming (that) z axis coincides with shaft axis, \( \sigma_z \) would be zero at shaft surface. Therefore, mean stress state in critical points of shaft surface act in a plane and mean stress state like Figure 6.

\[ (\tau_{x,z})_m = (\sigma_z)_m \]

Figure 6 Mean stress state on shaft

Considering bending and torsional alternating stresses, total alternating stress state is like Figure 7.

\[ (\tau_{\rho,\theta})_m = (\sigma_\rho)_m \]

Figure 7 Alternating stress state on shaft

Then using Mohr’s circle, principal alternating and mean stresses can be calculated from following Equations:
\[ (S_{1,2})_u = \frac{1}{2} \left( (\sigma_z)_u + (\tau_{x,z})_u \right) \]
\[ (S_{1,2})_m = \frac{(\sigma_z)_m + (\sigma_\rho)_m}{2} \pm \sqrt{\left( (\sigma_z)_u - (\sigma_\rho)_u \right)^2 + (\tau_{x,z})_u^2} \]

Now mutiaxial stress state is determined and fatigue analysis can be carried out in critical points.

4 Multiaxial fatigue analysis using stress based criteria

In fatigue calculation, uniaxial models are not accurate enough especially when crack density is high in material. Mutiaxial fatigue criteria reduce complex mutiaxial fatigue loading to a one uniaxial equivalent loading, which can be used to estimate fatigue life from S-N curves. Because loads act in different direction, mutiaxial fatigue analysis should be carried out and as the stresses amplitudes are small in compare with fatigue endurance limit, high cycle fatigue stress based criteria should be used.

Stress based criteria include variety of criteria proposed and still completing. In this paper the most common stress criteria are used. The first criteria are based on static yield criteria e.g. maximum normal stress, maximum shear stress and maximum octahedral stress. Results from equivalent stress criteria are compared with results of Sinès method.

5 Multiaxial fatigue analysis using equivalent static yield criteria extended to fatigue

Primary mutiaxial stress based criteria, are static yield criteria extended to fatigue. In fatigue calculation, stress components are replaced with amplitudes which are shown in Equations (10), (11) and (12) for maximum principal stress, maximum shear stress criteria and maximum octahedral stress criteria respectively [15 & 16],

\[ S_{qu} = S_{a1} \]
\[ S_{qu} = S_{a1} - S_{a3} \]
\[ S_{qu} = \frac{1}{2} \sqrt{(S_{a1} - S_{a2})^2 + (S_{a2} - S_{a3})^2 + (S_{a3} - S_{a1})^2} \]

in which \( S_{a1}, S_{a2}, S_{a3} \) are principal alternating stresses. Finding equivalent alternating stresses, equivalent mean stress can be calculated in two ways. First one is mean octahedral stress and second one is sum of mean normal stresses, which are shown in equation (13) and (14) respectively

\[ S_{qm} = \frac{1}{2} \sqrt{(S_{m1} - S_{m2})^2 + (S_{m2} - S_{m3})^2 + (S_{m3} - S_{m1})^2} \]
\[ S_{qm} = S_{m1} + S_{m2} + S_{m3} = S_{mx} + S_{my} + S_{mc} \]

in which \( S_{m1}, S_{m2}, S_{m3} \) are principal mean stresses. As Equation (13) incorrectly considers the effect of mean compressive stress, Equation (14) is used here. Finding equivalent mean stress and stress amplitude, it can be used with Goodman criterion to consider the mean stress effect

\[ S_{Nf} = \frac{S_{qu} S_u}{S_u - S_{qu}} \]

in which \( S_{Nf} \) is fatigue strength, \( S_u \) is ultimate strength and \( K_f \) are fatigue strength correction factors, e.g. surface finish and size factors. Finding fatigue strength it can be used with Basquin equation to find fatigue life [21] in the following form:

\[ N_f = \frac{1}{2} \left( \frac{S_{Nf}}{b} \right)^{1/2} \]

in which \( b \) is of fatigue strength exponent and \( \sigma_f \) is fatigue strength coefficient. Considering 1 million cycles, fatigue limit can be obtained. Then obtained fatigue limit should be modified with fatigue strength correction.
factors \( S'_{nf} = S_{nf} / \prod_{i=1}^{n} K_i \), and modified fatigue strength exponent is obtained using Equation (17)
\[
b' = \frac{\log(S'_{nf}) - \log(\sigma_f')}{\log(2 \times 10^6) - \log(1)}
\]
in this way Basquin equation is modified in each point to predict the fatigue life of complex multiaxial loading.

### 6 Sines method

Sines method uses octahedral shear stress amplitude as fatigue damage and mean hydrostatic stress as mean stress in the following form [2]:
\[
\Delta \tau_{oct}/2 + m(3\sigma_s) = \beta
\]
where \( m \) and \( \beta \) are determined by simple uniaxial tests. Substituting stress components, this criterion can be used to evaluate fatigue strength with Equation 19,
\[
\begin{align*}
\sqrt{(S_{mx} - S_{my})^2 + (S_{my} - S_{m})^2 + (S_{m} - S_{mx})^2} + 6(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2) \\
+ m(S_{mx} + S_{my} + S_{m}) = \sqrt{2} S_{nf}'
\end{align*}
\]

9)

Here, the \( m \) parameter is assumed 0.5 like many other steels. Again using Basquin equation, fatigue life can be calculated.

### 7 Results and discussion

Using the procedure discussed above, a MATLAB code is written to calculate the results in each critical point. Calculation is for VCN200 steel with material properties of \( S_n = 1090(MPa) \), \( \sigma_f' = 1780(MPa) \) and \( b = -0.115 \). For one million cycles, using strength correction factors, fatigue strengths obtained for various critical points on the shaft which are shown in Table 1.

#### Table 1 Fatigue strength of different points on the shaft [MPa]

<table>
<thead>
<tr>
<th>Point A</th>
<th>Point B</th>
<th>Point C</th>
<th>Point D</th>
<th>Point E</th>
</tr>
</thead>
<tbody>
<tr>
<td>Modified fatigue limit</td>
<td>103.8</td>
<td>145.7</td>
<td>88.57</td>
<td>217.16</td>
</tr>
<tr>
<td>Modified fatigue strength exponent</td>
<td>-0.1996</td>
<td>-0.1762</td>
<td>-0.2105</td>
<td>-0.1847</td>
</tr>
</tbody>
</table>

Results of fatigue strength calculation using different criteria are shown in Table 2.

#### Table 2 Fatigue strength obtained from different criteria [MPa]

<table>
<thead>
<tr>
<th>Point A</th>
<th>Point B</th>
<th>Point C</th>
<th>Point D</th>
<th>Point E</th>
</tr>
</thead>
<tbody>
<tr>
<td>Maximum principal stress</td>
<td>4.17</td>
<td>2.29</td>
<td>0.57</td>
<td>0.41</td>
</tr>
<tr>
<td>Maximum shear stress</td>
<td>7.59</td>
<td>3.53</td>
<td>0.86</td>
<td>0.55</td>
</tr>
<tr>
<td>Maximum octahedral stress</td>
<td>6.6</td>
<td>3.1</td>
<td>3.075</td>
<td>0.49</td>
</tr>
<tr>
<td>Sines method</td>
<td>9.93</td>
<td>6.34</td>
<td>20.412</td>
<td>7.52</td>
</tr>
</tbody>
</table>

Although all fatigue strengths are lower than predicted fatigue limit and fatigue failure may not occur theoretically, but because of high rotational speed of rotor which may cause early failure, fatigue lives are evaluated and analysed in critical points on the shaft surface. Table 3 shows the results of fatigue life prediction using obtained fatigue strength and Basquin equation.

#### Table 3 Predicted fatigue life using different criteria [cycle]

<table>
<thead>
<tr>
<th>Point A</th>
<th>Point B</th>
<th>Point C</th>
<th>Point D</th>
<th>Point E</th>
</tr>
</thead>
<tbody>
<tr>
<td>Maximum principal stress</td>
<td>1×10^{11}</td>
<td>2×10^{10}</td>
<td>3×10^{16}</td>
<td>2×10^{24}</td>
</tr>
<tr>
<td>Maximum shear stress</td>
<td>5×10^{10}</td>
<td>1×10^{11}</td>
<td>3×10^{15}</td>
<td>3×10^{23}</td>
</tr>
<tr>
<td>Maximum octahedral stress</td>
<td>9×10^{11}</td>
<td>3×10^{15}</td>
<td>6×10^{15}</td>
<td>6×10^{21}</td>
</tr>
<tr>
<td>Sines method</td>
<td>1×10^{11}</td>
<td>5×10^{15}</td>
<td>1×10^{9}</td>
<td>6×10^{15}</td>
</tr>
</tbody>
</table>

It is noted in Ref. [20] that at a stress level 2-3% lower than fatigue limit, crack initiation is not detected. But because of high frequency have positive effect on fatigue life [16], so there is no probability of fatigue failure using equivalent static yield criteria extended to fatigue. These criteria also show that point A is the weakest, because of lack of appropriate fillet due to design limitations and smaller diameter and redesign is necessary.

Sines method shows Point E on the mid cross-section plane on the hole surface as critical point. Because mean stresses are high at hole surface due to centrifugal stress concentration, the stress ratio related to crack initiation plane \( \rho = \sigma_{n, max}/\tau_a \), is far more than limiting value of \( \rho \) investigated in Ref. [17]. So failure mechanism approaches to static yield mechanism and the result should be verified.

It is shown in Ref. [18] that fatigue life prediction is conservative within factor 2 for in-phase or out of phase loading using octahedral stress criterion. But maximum shear stress criterion is conservative within factor 5 and finally Sines method shows less than 30 percent error in various kinds of loadings [19] under limiting \( \rho \) parameter. Because the similar test results have not found for high speed shaft, the validity of procedure and results have been checked with tension torsion SAE experimental data for long life (>10^6) available in Ref. [4] for the 1045 HR steel with \( \sigma'_{f} = 1043(MPa) \) and \( b = -0.111 \) which are shown in Table 4 for two maximum lives obtained in that research.

#### Table 4 SAE available long life test data

<table>
<thead>
<tr>
<th>( \sigma_n )</th>
<th>( \tau_a )</th>
<th>( \sigma'_{f} )</th>
<th>( N_{f, ext} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case 1</td>
<td>240</td>
<td>0</td>
<td>120</td>
</tr>
<tr>
<td>Case 2</td>
<td>0</td>
<td>152</td>
<td>0</td>
</tr>
</tbody>
</table>
Finally the validity check of G HOUGH, H POLLARD. The strength of metals under repeated stress with super imposed static stresses. Tech note 3495 National advisory committee for aeronautics, Washington, DC, 1955: 69. results obtained from this procedure and written code, are in reasonable order.

8 Conclusion
Fatigue strength and life prediction on potential critical points of shaft of a high-speed microgenerator rotor were theoretically investigated. Because the fatigue strength in all points are less than fatigue limit for one million cycle, failure is not expected; but as the shaft rotates in high speed, high number of cycles occur in some few hours. So noting to Table 3, redesign is necessary in point A to reach infinite life. Equivalent stress criteria result in almost the same life order. But Sines method yields different results, especially in point E because of the way it considers mean stress effect, as a parameter directly summed up to the fatigue damage. Life prediction by sines method in point E is not acceptable for the effect of mean stress that should be verified when the ratio of normal to shear stress is more than limiting value of $\rho$. Finally the validity check of the procedure with SEA experimental test data shows that results are in acceptable order. Therefore the results obtained here for multiaxial fatigue life prediction of high-speed microgenerator rotor, can be relied on.

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References

Table 5 Predicted life using MATLAB code and loading test data of Ref. [4]

<table>
<thead>
<tr>
<th>Case 1</th>
<th>Case 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Maximum principal stress 1.35*10^7</td>
<td>7.15*10^7</td>
</tr>
<tr>
<td>Maximum shear stress 1.14*10^7</td>
<td>8.89*10^6</td>
</tr>
<tr>
<td>Maximum octahedral stress 1.2*10^7</td>
<td>8.02*10^6</td>
</tr>
<tr>
<td>Sines method 1.01*10^7</td>
<td>1.77*10^6</td>
</tr>
</tbody>
</table>

1 So noting the scattered nature of fatigue data and statistical nature of fatigue life predictions, the results obtained from this procedure and written code, are in reasonable order.

2 Table 5 Predicted life using MATLAB code and loading test data of Ref. [4]

3 Case 1: 1.35*10^7, 7.15*10^7

4 Case 2: 1.14*10^7, 8.89*10^6

5 Maximum octahedral stress 1.2*10^7, 8.02*10^6

6 Sines method 1.01*10^7, 1.77*10^6

7 So noting the scattered nature of fatigue data and statistical nature of fatigue life predictions, the results obtained from this procedure and written code, are in reasonable order.

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Fatigue strength and life prediction on potential critical points of shaft of a high-speed microgenerator rotor were theoretically investigated. Because the fatigue strength in all points are less than fatigue limit for one million cycle, failure is not expected; but as the shaft rotates in high speed, high number of cycles occur in some few hours. So noting to Table 3, redesign is necessary in point A to reach infinite life. Equivalent stress criteria result in almost the same life order. But Sines method yields different results, especially in point E because of the way it considers mean stress effect, as a parameter directly summed up to the fatigue damage. Life prediction by sines method in point E is not acceptable for the effect of mean stress that should be verified when the ratio of normal to shear stress is more than limiting value of $\rho$. Finally the validity check of the procedure with SEA experimental test data shows that results are in acceptable order. Therefore the results obtained here for multiaxial fatigue life prediction of high-speed microgenerator rotor, can be relied on.

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