Math 285.001, Exam 3, November 17, 2004

Name: Solution Key

For full credit show all work and write the final answer in the correct place given for the answer.

Following is the list of matrices that have appeared in this test along with their eigen values and eigen vectors. In case of complex eigen values, note that the quantities in your answer are still expected to be real numbers.

1. For the matrix \( \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \) the eigen values are:
   
   -1 with eigen vector \( \begin{pmatrix} -1 \\ 1 \end{pmatrix} \) and
   5 with eigen vector \( \begin{pmatrix} 1 \\ 2 \end{pmatrix} \).

2. For \( \begin{pmatrix} -6 & 5 \\ -5 & 4 \end{pmatrix} \) the eigen value is -1, with multiplicity 2, and all eigen vectors are multiples of the vector \( \begin{pmatrix} 1 \\ 1 \end{pmatrix} \).

3. For \( \begin{pmatrix} 1 & -8 \\ 1 & -3 \end{pmatrix} \) the eigen values are:

   -1 + 2i with eigen vector \( \begin{pmatrix} 2 + 2i \\ 1 \end{pmatrix} \) and
   -1 - 2i with eigen vector \( \begin{pmatrix} 2 - 2i \\ 1 \end{pmatrix} \).
1. Find a particular solution to the following non-homogeneous differential equation using the method of undetermined coefficients:
\[ y'' + y' - 6y = 2x. \]  

Try \( y = ax + b, \)

\( y' = a \) and \( y'' = 0. \)

Plugging these in the equation give

\[ a - 6ax - 6b = 2x. \]

It follows that \(-6a = 2 \Rightarrow a = -1/3\) and

\[ a - 6b = 0 \Rightarrow b = -1/18. \]

\textbf{Answer:} \( y = -\frac{1}{3}x - \frac{1}{18} \)
2. Find the general solution to the differential equation using variation of parameters:
\[ y'' + y = \sec x. \] 15pts

The associated homogeneous equation is: \[ y'' + y = 0. \]
The auxiliary equation is: \[ m^2 + 1 = 0. \]
Solving this gives: \[ m = \pm i. \]
So the general solution to the associated homogeneous equation is:
\[ c_1 \cos x + c_2 \sin x. \]
The Wronskian \[ W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x - \sin^2 x = 1. \]

Let a particular solution to the nonhomogeneous equation be \[ y = u_1 \cos x + u_2 \sin x. \]
By variation of parameters, we have the formulas for \[ u'_1 = \frac{-y_2 f(x)}{W} = -\sin x / \cos x \]
and \[ u'_2 = \frac{y_1 f(x)}{W} = \cos x \sec x = 1. \]
So \[ u_1 = \int (-\sin x / \cos x) \, dx = \ln |\cos x|, \]
using the substitution \[ z = \cos x, \, dz = -\sin x \, dx. \]
\[ u_2 = \int 1 \, dx = x. \]

\[ \text{Answer: } y = c_1 \cos x + c_2 \sin x + \cos x \ln |\cos x| + x \sin x \]
3. Solve the following systems of equations. 10pts each

(a)

\[ \frac{dx}{dt} = x + 2y \]
\[ \frac{dy}{dt} = 4x + 3y \]

subject to: \( X(0) = \begin{pmatrix} 4 \\ 6 \end{pmatrix} \).

This is the system \( X' = AX \), with \( A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \).

As given on page 1, its eigenvalues are:

-1 with eigen vector \( \begin{pmatrix} -1 \\ 1 \end{pmatrix} \) and

5 with eigen vector \( \begin{pmatrix} 1 \\ 2 \end{pmatrix} \).

So the general solution is: \( c_1 e^{-t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 e^{5t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \).

From the initial value we have: \( c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \end{pmatrix} \).

This leads to:

\[ -c_1 + c_2 = 4 \]
\[ c_1 + 2c_2 = 6 \]

\( \Rightarrow c_1 = -2/3, \ c_2 = 10/3. \)

\[ \text{Answer: } X = \frac{-2}{3} e^{-t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \frac{10}{3} e^{5t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \]
(b)

\[
\frac{dx}{dt} = -6x + 5y \\
\frac{dy}{dt} = -5x + 4y
\]

This is the system \( X' = AX \), with \( A = \begin{pmatrix} -6 & 5 \\ -5 & 4 \end{pmatrix} \).

As given on page 1, its eigen value is \(-1\) with multiplicity 2, and an eigen vector \( \begin{pmatrix} 1 \\ 1 \end{pmatrix} \).

So one solution is: \( e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix} \).

To find another linearly independent solution we need to find a vector \( P \) such that \( (A - (-1)I)P = K \), where \( I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \).

\[
\begin{pmatrix} -5 & 5 \\ -5 & 5 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow
\]

\[
-5p_1 + 5p_2 = 1 \\
-5p_1 + 5p_2 = 1
\]

Letting \( p_1 = 0 \) gives us \( p_2 = 1/5 \).

So a second linearly independent solution is: \( te^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + e^{-t} \begin{pmatrix} 0 \\ 1/5 \end{pmatrix} \).

\[
\text{Answer: } X = c_1 e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \left[ te^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + e^{-t} \begin{pmatrix} 0 \\ 1/5 \end{pmatrix} \right]
\]
This is the system $X' = AX$, with $A = \begin{pmatrix} 1 & -8 \\ 1 & -3 \end{pmatrix}$.

As given on page 1 the eigen values are:

$-1 + 2i$ with eigen vector $\begin{pmatrix} 2 + 2i \\ 1 \end{pmatrix}$ and

$-1 - 2i$ with eigen vector $\begin{pmatrix} 2 - 2i \\ 1 \end{pmatrix}$.

So we have $\alpha = -1$, $\beta = 2$, $B_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, and $b_2 = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$.

Two linearly independent real solutions are:

$X_1 = e^{-t}[B_1 \cos 2t - B_2 \sin 2t]$ and $X_2 = e^{-t}[B_2 \cos 2t + B_1 \sin 2t]$.

The general solution is $c_1X_1 + c_2X_2$.

**Answer:** $X = c_1 e^{-t} \begin{pmatrix} 2 \cos 2t - 2 \sin 2t \\ \cos 2t \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 2 \cos 2t + 2 \sin 2t \\ \sin 2t \end{pmatrix}$. 
4. Use the definition of the Laplace transform of a function to find the Laplace transform of the piecewise defined function given below.  \[ f(t) = \begin{cases} 0, & 0 \leq t < \pi/2 \\ \sin t, & t \geq \pi/2 \end{cases} \]

\[ \mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}dt = \int_{\pi/2}^\infty \sin(t)e^{-st}dt. \]

We find \( I = \int \sin(t)e^{-st}dt \) using integration by parts twice and then solving for \( I \).

\[ \int \sin(t)e^{-st}dt = \frac{1}{1+s^2} (-e^{-st}\cos t - se^{-st}\sin t). \]

So \( \int_{\pi/2}^\infty \sin(t)e^{-st}dt = \left. \frac{1}{1+s^2} (-e^{-st}\cos t - se^{-st}\sin t) \right|_{\pi/2}^{\infty} = se^{-s\pi/2}/(1+s^2). \]

**Answer:** \( \mathcal{L}\{f(t)\} = se^{-s\pi/2}/(1+s^2) \)
5. Use the formulas we learned and the linearity of the Laplace transform to find each of the following. 5pts each

(a) \( \mathcal{L}\{3t + 5\} \)

\[ \mathcal{L}\{3t + 5\} = 3\mathcal{L}\{t\} + 5\mathcal{L}\{1\} = \frac{3}{s^2} + \frac{5}{s}. \]

**Answer:** \( \frac{3}{s^2} + \frac{5}{s} \)

(b) \( \mathcal{L}\{\cos(3t) - 4\sin(2t)\} \)

\[ \mathcal{L}\{\cos(3t) - 4\sin(2t)\} = \mathcal{L}\{\cos(3t)\} - 4\mathcal{L}\{\sin(2t)\} = \frac{s}{s^2 + 9} - 4\frac{2}{s^2 + 4}. \]

**Answer:** \( \frac{s}{s^2 + 9} - \frac{8}{s^2 + 4} \).
6. Find the following inverse Laplace transforms. 10pts each

(a) \( \mathcal{L}^{-1}\left\{ \frac{1}{s^2} + \frac{1+s}{s^2+49} \right\} \)

\[
\mathcal{L}^{-1}\left\{ \frac{1}{s^2} + \frac{1+s}{s^2+49} \right\} = \mathcal{L}^{-1}\left\{ \frac{1}{s^2} \right\} + \mathcal{L}^{-1}\left\{ \frac{1}{s^2+49} \right\} + \mathcal{L}^{-1}\left\{ \frac{s}{s^2+49} \right\} = \frac{t^2}{2} + \frac{1}{7} \sin 7t + \cos 7t.
\]

**Answer:** \( \frac{t^2}{2} + \frac{1}{7} \sin 7t + \cos 7t. \)

(b) \( \mathcal{L}^{-1}\left\{ \frac{s}{(s-2)(s-3)} \right\} \)

By partial fractions we have,
\[
\frac{s}{(s-2)(s-3)} = \frac{4}{s-2} + \frac{B}{s-3}, \quad \text{and} \quad A = -2, \quad B = 3.
\]
So \( \mathcal{L}^{-1}\left\{ \frac{s}{(s-2)(s-3)} \right\} = \mathcal{L}^{-1}\left\{ \frac{-2}{s-2} \right\} + \mathcal{L}^{-1}\left\{ \frac{3}{s-3} \right\} = -2e^{2t} + 3e^{3t}. \)

**Answer:** \( -2e^{2t} + 3e^{3t}. \)