(1) Fill in the blanks in the statement of the Squeeze Theorem below:
Suppose that \( \{x_n\}, \{y_n\} \) and \( \{z_n\} \) are sequences of real numbers.
If \( x_n \to a \) and \( y_n \to a \) as \( n \to \infty \) and if there is an \( N_0 \in \mathbb{N} \) such that \( x_n \leq z_n \leq y_n \) for \( n \geq N_0 \), then \( z_n \to a \) as \( n \to \infty \). 3pts

(2) Using the definition show that the sequence \( \{-n^2 + 1\} \) diverges to \(-\infty\) as \( n \to \infty \). 3pts

Let \( M \in \mathbb{R} \) be given. As all terms of the above sequence are non-positive, without loss of generality assume that \( M < 0 \). By the Archimedes property, there is a natural number \( N \) such that
\[
N > \sqrt{-M+1} \implies N^2 > -M + 1 \implies -N^2 < M - 1 \implies 1 - N^2 < M.
\]
It is easy to see that \( \forall n \geq N, \text{ we have } 1 - n^2 < M. \)
Therefore by definition, the sequence \( \{-n^2 + 1\} \) diverges to \(-\infty\) as \( n \to \infty \).

(3) Circle True or False and give a one-line justification, such as the name of the theorem used or a brief counter example without proof. 2pts each
(a) True / False
If \( \{x_n\} \) is a strictly decreasing sequence and \( 0 \leq x_n < 1 \) for all \( n \), then \( x_n \to 0 \) as \( n \to \infty \).

Counter example: Consider \( x_n = \frac{1}{2} + \frac{1}{10n} \). Then \( \{x_n\} \) is a strictly decreasing sequence and \( 0 \leq x_n < 1 \) for all \( n \), however, \( x_n \to \frac{1}{2} \) as \( n \to \infty \).

(b) True / False
The sequence \( \frac{\cos(n^2+2n+5)}{3} \) has a convergent subsequence.

This sequence is bounded below by \(-\frac{1}{3}\) and bounded above by \(\frac{1}{3}\). By Bolzano-Weierstrass Theorem it has a convergent subsequence.