2.87) Define \( f : [0, 1] \to \mathbb{R} \) by

\[
f(x) = \begin{cases} 
0, & \text{if } x = 0 \text{ or } x \in [0, 1] - \mathbb{Q}; \\
1/q & \text{if } x \in (0, 1] \cap \mathbb{Q} \text{ and } x = p/q \text{ in lowest terms}.
\end{cases}
\]

(a) Show that the set of points of discontinuity of \( f \) is \((0, 1] \cap \mathbb{Q}\).

Proof: First let \( \epsilon > 0 \) and \( x = \frac{p}{q} \in (0, 1] \cap \mathbb{Q} \) be given. Then with \( f \) as above we have that \( f\left(\frac{p}{q}\right) = \frac{1}{q} \).

We also have the following inequality

\[
\frac{1}{q + 1} < \frac{1}{q} < \frac{1}{q - 1}
\]

Now take \( \epsilon \) such that

\[
\epsilon < \frac{1}{2} \left| \frac{1}{q + 1} - \frac{1}{q} \right|
\]

Now consider the interval \((x - \delta, x + \delta)\) and by the density of the irrational numbers it follows that there exists some \( y \in [0, 1] - \mathbb{Q} \) such that \( y \in (x - \delta, x + \delta) \). Then we have \( |x - y| < \delta \), but

\[
|f(x) - f(y)| = \frac{1}{q} - 0 > \epsilon
\]

Therefore we may conclude that the points \( x \in (0, 1] \cap \mathbb{Q} \) are the points of discontinuity.

Alternative proof: Let \( x_0 = \frac{p}{q} \in \mathbb{Q} \cap (0, 1] \). Let \( \epsilon = \frac{1}{2q} \). For any \( \delta > 0 \), there is an irrational \( y \) in the interval \((x - \delta, x + \delta)\) by the density of irrational numbers. Then we have \( |x - y| < \delta \), and

\[
|f(x_0) - f(y)| = \frac{1}{q} - 0 > \epsilon
\]

It follows that \( f \) is not continuous at \( x_0 \).

Next, we want to show that \( f \) is continuous at 0. Let \( \epsilon > 0 \) be given. Take \( \delta = \epsilon \).

If \( x \in [0, \delta) \) is an irrational, \( |f(x) - f(0)| = |0 - 0| = 0 < \delta \).

If \( x \in [0, \delta) \) is a rational, say \( \frac{p}{q} \), and \( f(x) = \frac{1}{q} \leq \frac{p}{q} = |x - 0| < \delta = \epsilon \).

Now I wish to show that these are the only points of discontinuity, i.e. the function is continuous at 0, and at every irrational number. So first let \( \epsilon > 0 \) be given and now I want to find a \( \delta \) such that if \( x \in (0, 1] \cap \mathbb{Q} \) and \( |x - x_0| < \delta \) then I will have \( |f(x) - f(x_0)| < \epsilon \). First let \( x \in [0, 1] \cap \mathbb{Q} \) be given and then take \( \frac{1}{n} < \epsilon \), which can be done by the Archimedean property. Now consider the following list of all rational numbers with denominator less than \( n \):

\[
\frac{1}{n-1} \quad \frac{2}{n-1} \quad \frac{3}{n-1} \quad \ldots \quad \frac{n-1}{n-1}
\]

\[
\vdots
\]

\[
\frac{1}{2} \quad \frac{2}{2}
\]

\[
\frac{1}{1}
\]
Now set \( A \) equal to this set of all rational numbers written in lowest terms with denominator less than \( n \). One other observation is that this set \( A \) is finite and thus we may set \( \delta = \min_{c \in A} \{ |x - c| \} \). Now by doing this we have that \( \forall c \in A, \) that \( c \notin (x - \delta, x + \delta) \). From this we can see that for any \( \frac{p}{q} \in (x - \delta, x + \delta), q \geq n \) and thus \( \frac{1}{q} \leq \frac{1}{n} \). Then we have that

\[
\left| f(x) - f\left( \frac{p}{q} \right) \right| = \frac{1}{q} \leq \frac{1}{n} < \epsilon.
\]

Now any other \( y \in (x - \delta, x + \delta) \) is either zero or irrational, but we still arrive at the result

\[
|f(x) - f(y)| = |0 - 0| = 0 < \epsilon
\]

So therefore \( f \) is not continuous on \((0, 1] \cap \mathbb{Q}, \) but is continuous on any other value on \([0, 1], \) which is what we wanted. QED

**Alternative proof that \( f \) is continuous at any irrational \( x_0 \in [0, 1]:**

Let \( \epsilon > 0 \) be given. By the Archimedean property, there exists an \( N \in \mathbb{N} \) such that \( N > \frac{1}{\epsilon} \). It follows that \( \frac{1}{N} < \epsilon \).

Now, let \( p \) be the largest integer less than or equal to \( Nx_0 \). Then \( \frac{p}{N} \leq x_0 \leq \frac{p+1}{N} \).

Let \( \delta = \min\{ x_0 - \frac{p}{N}, \frac{p+1}{N} - x_0 \} \).

Claim: \( |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon \). Clearly, if \( |x - x_0| < \delta, \) then \( \frac{p}{N} \leq x \leq \frac{p+1}{N} \).

This implies that either \( x \) is an irrational, or \( x = \frac{a}{q} \) with \( q > N \). So \( f(x) \) is either 0, or \( \frac{1}{q} < \frac{1}{N} < \epsilon \).

(b) Deduce from part(a) that \( f \) is Riemann integrable on \([0, 1].\)

Proof: From part (a) we can see that the set of discontinuities is the rational numbers and since this set is countable we may conclude from a proof in class that it has measure zero. Now the function is also bounded on \([0, 1]\) because for any \( \frac{p}{q}, f\left( \frac{p}{q} \right) = \frac{1}{q} \leq 1 \). Therefore by theorem 2.7 our function is Riemann integrable.

(c) Show that \( \int_0^1 f(x)dx = 0.\)

First consider the lower Riemann integral \( \int_0^1 f(x)dx = \sup \left\{ \int_0^1 h(x)dx : \text{h is a step function and } h \leq f \right\} \).

Note that a step function is given by \( h(x) = \Sigma_{k=1}^n a_k \chi_{I_k} \). A nondegenerate interval contains irrationals, so on a nondegenerate interval \( I_k, \) in order for \( h(x) \) to be dominated by \( f(x), \) we need \( a_k \leq 0. \) As there can be only finitely many degenerate intervals, \( h(x) \) can be greater than 0 at only finitely many points.

It follows that \( \sup \left\{ \int_0^1 h(x)dx : \text{h is a step function and } h \leq f \right\} = 0 \Rightarrow \int_0^1 f(x)dx = 0. \)

2.90) Refer to Exercise 2.89. Is it possible to find such a sequence of functions if the sequence is required to be monotone?

No, it’s not possible to find such a sequence. Dini’s Theorem states that if we have a monotone sequence of continuous functions \( \{ f_n \}_{n=1}^\infty \) defined on a closed bounded interval \([a, b] \) which converges pointwise to the continuous function \( f, \) then \( f_n \to f \) uniformly on \([a, b]. \) Now this sequence \( \{ f_n \}_{n=1}^\infty \) described above of continuous functions will be Riemann integrable because they are continuous and they also converge uniformly by Dini’s theorem. Thus by theorem 2.8 the limit and integral can be interchanged.