Show that the Cantor set, \( P \), has Lebesgue outer measure zero?

**Show:**

By definition, the Cantor set is a subset of \([0, 1]\) obtained by successive deletion of the middle third open intervals of \([0, 1]\).

\[
P \subset P_n = \bigcup_{k=1}^{2^n} P_{n,k}, 1 \leq k \leq n
\]

\( P \) is the union of \( 2^n \) disjoint closed intervals each with the length \( \frac{1}{3^n} \). Let \( \epsilon > 0 \) be given, then since 
\[
\lim_{n \to \infty} \left( \frac{1}{3} \right)^n = 0, \exists n \in \mathbb{N} \quad \exists \left( \frac{1}{3} \right)^n < \frac{\epsilon}{2}.
\]

Now, we make an adjustment to get open intervals of total length less than \( \epsilon \) that contain \( P \). Such adjustment could be done by replacing each \( P_{n,k} = [a_k, b_k] \) with \( I_{n,k} = (a_k - \delta, b_k + \delta) \) where \( \delta < \frac{\epsilon}{2} \).

An immediate result of the above adjustment is:

\[
P \subset \bigcup_{k=1}^{2^n} I_{n,k}, \text{ each } I_{n,k} \text{ is an open interval}
\]

Since,

\[
\lambda^*(p) = \inf \left\{ \sum_{k=1}^{2^n} \ell(I_{n,k}) : \{I_{n,k}\}_k, \text{ open interval, } P \subset \bigcup_{k} I_{n,k} \right\}
\]

then, by definition of infimum, we have:

\[
\lambda^*(p) \leq \sum_{k=1}^{2^n} \ell(I_{n,k}) = 2^n(2\delta) + \left( \frac{2}{3} \right)^n \epsilon < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\]

But, since \( \epsilon \) was arbitrary, we get \( \lambda^*(p) = 0. \)
For $A \subset \mathcal{R}$ and $b \in \mathcal{R}$, define $bA = \{ba : a \in A\}$. Show that $\lambda^*(bA) = |b|\lambda^*(A)$?

**Lemma 1:** Given any finite open interval $I = (x, y)$ and $x, y, b \in \mathcal{R}$ then, $\ell(bI) = |b|\ell(I)$.

**proof:**

$$
\ell(bI) = |by - bx| = |b(y - x)| = |b||y - x| = |b| \ell(I).
$$

**Lemma 2:** $A \subset \bigcup_{n=1}^{\infty} I_n \Leftrightarrow bA \subset \bigcup_{n=1}^{\infty} bI_n$, $b \neq 0$.

**proof:**

$\Rightarrow$ suppose $A \subset \bigcup_{n=1}^{\infty} I_n$ and let $x \in bA$. Then, by definition of $bA$, there exists $a \in A$ such that $x = ba$. Now, since $a \in A$ and $A \subset \bigcup_{n=1}^{\infty} I_n$, it follows that $a \in \bigcup_{n=1}^{\infty} I_n$ so, $a \in I_n$ for some $I_n$ so that $ba \in bI_n \subset b\bigcup_{n=1}^{\infty} I_n$. That is, $x \in \bigcup_{n=1}^{\infty} bI_n$. Hence $A \subset \bigcup_{n=1}^{\infty} I_n \Rightarrow bA \subset \bigcup_{n=1}^{\infty} bI_n$.

$\Leftarrow$ suppose $bA \subset \bigcup_{n=1}^{\infty} bI_n$. Then, $(\frac{1}{n})ba \subset \bigcup_{n=1}^{\infty} bI_n$. Hence $bA \subset \bigcup_{n=1}^{\infty} bI_n \Rightarrow A \subset \bigcup_{n=1}^{\infty} I_n$.

**Lemma 3:** For a real number $\eta$, define $\eta A = \{\eta x : x \in A\}$. If $A$ is a bounded subset of $\mathcal{R}$, then $\eta A$ is bounded and $\inf \eta A = \eta \inf A$ whenever $\eta > 0$

**proof:**

Since $A$ is bounded, $\exists k \in \mathcal{R}$ s.t. $k \geq |x|$ $\forall x \in A$. Therefore we have, $|\eta x| = |\eta||x| \leq |\eta|k$ $\forall x \in \eta A$ which implies that $\eta A$ is bounded. Let $\ell = \inf A$ then by definition of infimum $x \geq \ell$ $\forall x \in A$. But, since $\eta > 0$ it follows that $\eta x \geq \eta \ell$ $\forall x \in \eta A$. Hence, $\eta \ell$ is a lower bound for $\eta A$ and therefore (\itm*)... $\inf \eta A = \eta \inf A$.

Now, since $\eta > 0$ by assumption, then for any $\epsilon > 0$ we have $\frac{\epsilon}{\eta} > 0$. That is, $\ell + \frac{\epsilon}{\eta}$ is not a lower bound for $A$ since $\ell + \frac{\epsilon}{\eta} > \ell$. Thus, $\exists x_1 \in A$ s.t. $x_1 < \ell + \frac{\epsilon}{\eta}$. Again, since $\eta > 0$ by assumption, then it follows that $\eta x_1 < \eta \ell + \epsilon$ where $\eta x_1 \in \eta A$. That is, any number greater than $\eta \ell$ cannot be a lower bound of $\eta A$. Hence, (\itm*)... $\inf \eta A \leq \eta \ell$. Finally, from (\itm*) and (\itm*) it follows that $\inf \eta A = \eta \ell = \eta \inf A$.

For the special case when $b = 0$ the proof is trivial, in fact $\lambda^*(\{0\}) = 0$ since all singletons have outer measure zero. But, for the general case when $b \neq 0$ we implement the proof as follows:

For each $E \subset \mathcal{R}$ let $S_E = \{\sum_n \ell(I_n) : \{I_n\}_n \text{open intervals, } \bigcup I_n \supset E\}$. Then, since $A \subset \bigcup_{n=1}^{\infty} \Leftrightarrow bA \subset \bigcup_{n=1}^{\infty} bI_n$ (from lemma 2) and $\ell(bI) = |b|\ell(I)$ (from lemma 1), we get:

$$S_{bA} = \sum_n \ell(bA) = |b| \sum_n \ell(I_n) = |b| S_A \quad (1)$$

But, by definition of Lebesgue outer measure we have:

$$(i)\ldots \lambda^*(A) = \inf S_A$$

$$(ii)\ldots \lambda^*(bA) = \inf S_{bA}$$

Therefore,

$$\lambda^*(bA) = \inf S_{bA} \quad \text{by (ii)}$$

$$= \inf |b| S_A \quad \text{by (1)}$$

$$= |b| \inf S_A \quad \text{by (lemma 3)}$$

$$= |b| \lambda^*(A) \quad \text{by (i)}$$

\[\square\]
proof of lemma 3.8: Let $I = (a, b)$ be a finite open interval where $a, b \in \mathbb{R}$, and let $\epsilon, \delta > 0$ be given. By the Archimedean principle $\exists n \in \mathbb{N} \ni \frac{b-a}{n} < \frac{\delta}{2}$. We divide $(a, b)$ to $n$ equal subintervals such that:

$J_1 = (a_1, b_1), J_2 = (a_2, b_2), \ldots, J_k = (a_k, b_k), 1 \leq k \leq n$

where $(a, b) = \bigcup_{k=1}^{\infty} (a_k, b_k) = (a_1, a_1 + \frac{\delta}{4}) \cup (a_1 + \frac{\delta}{4}, a_1 + \frac{3\delta}{4}) \cup \ldots \ldots$ and $\sigma = \min\{2\delta, \frac{\delta}{n}\}$

$\ell(J_k) < \delta$ since $\frac{b-a}{n} < \frac{\delta}{2}$ and $1 \leq k \leq n$

$\bigcup_{k=1}^{n} J_k \supset I$ since $J_i \cap J_j \neq \emptyset \ \forall j = i + 1$

From the way our $J$’s were constructed, it’s obvious that $\sum \ell(J_k)$ will exceed $\ell(I)$ by the total length of the overlapped edges of all $J$’s. That is,

$$\sum_{k=1}^{n} \ell(J_k) = \ell(I) + \Psi \quad \text{where } \Psi \text{ equals the total length of the overlapped edges of the intervals } J_k$$

But since $\Psi = n \frac{\delta}{4} < n \frac{\delta}{4n} = \epsilon$, it follows that:

$$\sum_{k=1}^{n} \ell(J_k) < \ell(I) + \epsilon$$

\[ \square \]