2.79 Let \( f \) be a bounded function on \([a, b]\). Prove that
\[
\int_a^b f(x) \, dx \leq \liminf_{n \to \infty} \int_a^b f_n(x) \, dx
\]

Hints: Use Exercise 2.78.
Let
\[
A = \left\{ \int_a^b h(x) \, dx : h \text{ a step function, } h \leq f \right\} \quad \text{and} \quad B = \left\{ \int_a^b h(x) \, dx : h \text{ a step function, } h \geq f \right\}.
\]
Then \( A, B \) are nonempty sets with \( \int_a^b f(x) \, dx = \sup A \) and \( \int_a^b f(x) \, dx = \inf B \). By Exercise 2.78 we have \( a \leq b \) for all \( a \in A \) and \( b \in B \). It remains to show that \( \sup A \leq \inf B \). Let \( b \in B \) be given; then \( b \) is an upper bound for \( A \) since \( a \leq b \) for all \( a \in A \). Hence, \( \sup A \leq b \). But since \( b \in B \) was arbitrary we have \( \sup A \leq b \) for all \( b \in B \). Thus, \( \sup A \) is a lower bound for \( B \) and therefore \( \sup A \leq \inf B \). □

2.83 Prove Proposition 2.21 on page 85: Let \( \{E_n\}_n \) be a sequence of subsets of \( \mathbb{R} \) each having measure zero. Then \( \bigcup_n E_n \) has measure zero.

Observe that a subset \( E \subset \mathbb{R} \) has measure zero iff for every \( \varepsilon > 0 \) there is a countable family of open intervals \( \{I_k\}_{k \in K} \) such that \( E \subset \bigcup_{k \in K} I_k \) and \( \sum_{k \in K} \ell(I_k) < \varepsilon \); usually \( K = \mathbb{N} \) but it may be any countable set. Now let \( \{E_m\}_m \) be a sequence of subsets of \( \mathbb{R} \) each having measure zero. For each \( m = 1, 2, \ldots \), since \( E_m \) has measure zero, there is a sequence of open intervals \( \{I_{m,n}\}_n \) such that \( E_m \subset \bigcup_n I_{m,n} \) and \( \sum_n \ell(I_{m,n}) < \frac{\varepsilon}{2^m} \). Hence we have a countable family of open intervals \( \{I_{m,n}\}_{m,n} \) (note that the indexing set is a countable union of countable sets and thus is itself countable) such that
\[
\bigcup_m E_m \subset \bigcup_m \bigcup_n I_{m,n} = \bigcup_{m,n} I_{m,n}
\]

and
\[
\sum_{m,n} \ell(I_{m,n}) = \sum_m \sum_n \ell(I_{m,n}) < \sum_m \frac{\varepsilon}{2^m} \leq \varepsilon.
\]
Hence, \( \bigcup_m E_m \) has measure zero. □

2.89 Construct a sequence of continuous functions on \([0, 1]\) that converges pointwise to a continuous function but for which the limit and the integral cannot be interchanged.

Let \( f \) be the constant function 0 and for each \( n \in \mathbb{N} \) define \( f_n : [0, 1] \to \mathbb{R} \) as follows:
\[
f_n(x) = \begin{cases} 
  n^2 x, & 0 \leq x \leq \frac{1}{2n}; \\
  n - n^2 x, & \frac{1}{2n} < x \leq \frac{1}{n}; \\
  0, & \frac{1}{n} < x \leq 1.
\end{cases}
\]
Then \( f_n \) is continuous for each \( n \in \mathbb{N} \) and we have \( f_n \to f \) pointwise. We have \( \int_0^1 f_n(x) \, dx = \frac{1}{4} \) and \( \int_0^1 f(x) \, dx = 0 \). Hence,
\[
\lim_{n \to \infty} \int_0^1 f_n(x) \, dx = \frac{1}{4} \neq 0 = \int_0^1 \lim_{n \to \infty} f_n(x) \, dx.
\]
Hence, the limit and the integral operations cannot be interchanged.