3.15 Prove part (d) of Proposition 3.1: \( \lambda^*(x + A) = \lambda^*(A) \) for \( x \in \mathbb{R}, A \subseteq \mathbb{R} \), where \( x + A = \{x + y : y \in A\} \).

Let \( x \in \mathbb{R} \) be given and let \( A \) be a subset of \( \mathbb{R} \); recall that \( \lambda^*(A) = \inf S_A \) where

\[
S_A = \left\{ \sum_n \ell(I_n) : \{I_n\}_n \text{ open intervals, } \bigcup_n I_n \supseteq A \right\}.
\]

To show that \( \lambda^*(x + A) = \lambda^*(A) \), it suffices to show that \( S_{x+A} = S_A \). Suppose that \( y \in S_A \); then there is a sequence of open intervals \( \{I_n\}_n \) such that \( A \subseteq \bigcup_n I_n \) and \( y = \sum_n \ell(I_n) \). Now set \( J_n = x + I_n \); then, for each \( n \), \( J_n \) is an open interval with the same length as \( I_n \) and we have \( x + A \subseteq \bigcup_n J_n \). Hence, \( y = \sum_n \ell(J_n) \in S_{x+A} \) and so \( S_A \subseteq S_{x+A} \); but the same argument shows that \( S_{x+A} \subseteq S_{-x+(x+A)} = S_A \). Therefore, \( S_{x+A} = S_A \). □

3.16 Let \( A \) be a set with \( \lambda^*(A) < \infty \). Show that the function, \( g \), defined by \( g(x) = \lambda^*(A \cap (-\infty, x]) \) is uniformly continuous on \( \mathbb{R} \).

Note that for every real number \( x \), \( g(x) \) is nonnegative and

\[
g(x) = \lambda^*(A \cap (-\infty, x]) \leq \lambda^*(A) < \infty
\]

by monotonicity; hence \( g \) is a real valued function. We will first show that \( |g(x) - g(y)| \leq |y - x| \) for all \( x, y \in \mathbb{R} \). Since \( g \) is nondecreasing it suffices to show that \( g(x) - g(y) \leq x - y \) for all \( x, y \in \mathbb{R} \) with \( x > y \). Let \( x, y \in \mathbb{R} \) be given with \( x > y \) and let \( \varepsilon > 0 \). By definition of \( g(y) \) there is a sequence of open intervals \( \{I_n\}_n \) such that \( A \cap (-\infty, y] \subseteq \bigcup_n I_n \) and \( \sum_n \ell(I_n) < g(y) + \frac{\varepsilon}{3} \). Set \( I_0 = (y - \frac{\varepsilon}{3}, x + \frac{\varepsilon}{3}) \); then we have \( (-\infty, x] \subseteq I_0 \cup (-\infty, y] \) and thus

\[
A \cap (-\infty, x] \subseteq I_0 \cup (A \cap (-\infty, y]) \subseteq I_0 \cup (\bigcup_n I_n).
\]

Hence,

\[
g(x) = \lambda^*(A \cap (-\infty, x]) \leq \ell(I_0) + \sum_n \ell(I_n) < y - x + \frac{2\varepsilon}{3} + g(y) + \frac{\varepsilon}{3} = y - x + g(y) + \varepsilon.
\]

So we have \( g(x) - g(y) < x - y + \varepsilon \) for all \( \varepsilon > 0 \); hence, \( g(x) - g(y) \leq x - y \). Therefore, we have \( |g(x) - g(y)| \leq |x - y| \) for all \( x, y \in \mathbb{R} \). It remains to show that \( g \) is uniformly continuous. Let \( \varepsilon > 0 \) and \( x \in \mathbb{R} \) be given. Then take \( \delta = \varepsilon \); if \( |y - x| < \varepsilon \), we have \( |g(x) - g(y)| \leq |x - y| < \varepsilon \) as required. Thus \( g \) is uniformly continuous. □

3.18 Let \( E \subseteq \mathbb{R} \). Show that there is a sequence of open sets, \( \{O_n\}_{n=1}^\infty \), such that \( O_1 \supseteq O_2 \supseteq \cdots \supseteq E \) and

\[
\lambda^*(E) = \lambda^*\left(\bigcap_{n=1}^\infty O_n\right) = \lim_{n \to \infty} \lambda^*(O_n).
\]

If \( \lambda^*(E) = \infty \) we may take \( O_n = \mathbb{R} \) for all \( n \in \mathbb{N} \) and the result holds; hence, we may suppose that \( \lambda^*(E) < \infty \).

Claim: For every \( \varepsilon > 0 \), there is an open set \( O \supset E \) such that \( \lambda^*(O) < \lambda^*(E) + \varepsilon \).

Let \( \varepsilon > 0 \) be given. Then by definition of \( \lambda^*(E) \) as infimum of \( S_E \), \( \lambda^*(E) + \varepsilon \) is not a lower bound of \( S_E \). Hence, there is a sequence of open intervals \( \{I_n\}_n \) such that \( E \subseteq \bigcup_n I_n \) and \( \sum_n \ell(I_n) < \lambda^*(E) + \varepsilon \). Set \( O = \bigcup_n I_n \); then since \( O \subseteq \bigcup_n I_n \), we have \( \sum_n \ell(I_n) \subseteq S_O \). Thus,

\[
\lambda^*(O) \leq \sum_n \ell(I_n) < \lambda^*(E) + \varepsilon.
\]

The claim now follows.

By the claim (and the Axiom of Choice) there is a sequence of open sets \( \{U_n\}_{n=1}^\infty \) such that for each \( n \in \mathbb{N} \) we have \( U_n \supset E \) and \( \lambda^*(U_n) < \lambda^*(E) + \frac{1}{n} \). Now for each \( n \in \mathbb{N} \) set \( O_n = \bigcap_{k=1}^n U_k \). Then for each \( n \in \mathbb{N} \) we have \( O_n \) is an open set, \( O_n \supset O_{n+1} \) and \( E \subseteq O_n \subseteq U_n \); so, for each \( m \in \mathbb{N} \) we have by monotonicity

\[
\lambda^*(E) \leq \lambda^*\left(\bigcap_{n=1}^\infty O_n\right) \leq \lambda^*(O_m) \leq \lambda^*(U_m) < \lambda^*(E) + \frac{1}{m}.
\]

Hence taking the limit as \( m \to \infty \) we get

\[
\lambda^*(E) = \lambda^*\left(\bigcap_{n=1}^\infty O_n\right) = \lim_{n \to \infty} \lambda^*(O_n).
\]

The result follows. □
Verify parts (a) and (b) in Lemma 3.10 on page 113:

Let $O$ be a proper open subset of $\mathbb{R}$ (i.e., $O$ is open, nonempty and not equal to $\mathbb{R}$). For each $n \in \mathbb{N}$, let

$$O_n = \left\{ x : d(x, O^c) > \frac{1}{n} \right\}.$$

Then,

a) $O_n$ is open and $O_n \subset O$ for all $n \in \mathbb{N}$.

Let $n \in \mathbb{N}$; then since $d(x, O^c) = 0$ for all $x \in O^c$ we have $O_n \subset O$. As was proved in class the function $g : \mathbb{R} \to \mathbb{R}$ given by $g(x) = d(x, O^c)$ is continuous. Therefore by Theorem 2.5 we have

$$O_n = \left\{ x : d(x, O^c) > \frac{1}{n} \right\} = g^{-1}\left( \frac{1}{n}, \infty \right)$$

is open (since $(\frac{1}{n}, \infty)$ is open).

b) $O_1 \subset O_2 \subset \cdots$ and $\bigcup_n O_n = O$.

Let $g$ be as in part (a). Let $n \in \mathbb{N}$ be given; since $\frac{1}{n+1} < \frac{1}{n}$, we have $(\frac{1}{n}, \infty) \subset (\frac{1}{n+1}, \infty)$ and therefore

$$O_n = g^{-1}\left( \frac{1}{n}, \infty \right) \subset g^{-1}\left( \frac{1}{n+1}, \infty \right) = O_{n+1}.$$  

Since $O^c$ is closed we have $x \in O^c$ iff $g(x) = 0$, and so $x \in O$ iff $g(x) > 0$. Let $x \in O$ be given; then since $g(x) > 0$ there is $n \in \mathbb{N}$ such that $g(x) > \frac{1}{n}$. Hence, $x \in O_n$ and so $O \subset \bigcup_n O_n$. The reverse inclusion follows by part (a).

3.25 Suppose that $O$ is open. Prove that

$$\lambda^*(W) = \lambda^*(W \cap O) + \lambda^*(W \cap O^c)$$

for all subsets $W$ of $\mathbb{R}$.

Let $W \subset \mathbb{R}$ be given and set $A = W \cap O$ and $B = W \cap O^c$. Then since $A \subset O$ and $B \subset O^c$ we have,

$$W = (W \cap O) \cup (W \cap O^c) = A \cup B$$

and

$$\lambda^*(W) = \lambda^*(A \cup B) = \lambda^*(A) + \lambda^*(B) = \lambda^*(W \cap O) + \lambda^*(W \cap O^c)$$

by Theorem 3.9.