71 Prove that Lemma 3.15 holds for all nonnegative $M$-measurable functions. That is, if $f$ is a nonnegative Lebesgue measurable function and $\{E_n\}_{n=1}^\infty$ is a sequence of Lebesgue measurable sets with $E_1 \subset E_2 \subset \cdots$, then

$$\int_{\bigcup_{n=1}^\infty E_n} f \, d\lambda = \lim_{n \to \infty} \int_{E_n} f \, d\lambda.$$ 

Set $E = \bigcup_{n=1}^\infty E_n$. We apply the Monotone Convergence Theorem (Theorem 3.17) to the sequence $\{f_n\}$ where $f_n = f|_{E_n}$. Since $f$ is a nonnegative Lebesgue measurable function the same is true for each $f_n$. Further for each $n$ we have $E_n \subset E_{n+1}$ so $f_n(x) \leq f_{n+1}(x)$ for all $x \in \mathbb{R}$. Finally, $\lim_{n \to \infty} f_n(x) = f(x)\chi_E(x) < \infty$ for all $x \in \mathbb{R}$. Hence, we have

$$\int_E f \, d\lambda = \int f\chi_E \, d\lambda = \lim_{n \to \infty} \int f_n \, d\lambda = \lim_{n \to \infty} \int f\chi_{E_n} \, d\lambda$$

by the Monotone Convergence Theorem (applied at (⋆)).

73 Provide an example where strict inequality holds in Fatou’s lemma and where $\lim_{n \to \infty} \int_E f_n \, d\lambda$ exists.

For each $n \in \mathbb{N}$ set $f_n = \chi_{[n,n+1]}$; then for each $x \in \mathbb{R}$ we have $f_n(x) \to 0$ (since if $n > x$, $f_n(x) = 0$). Note that $\int f_n \, d\lambda = 1$ for all $n$ and so $\lim_{n \to \infty} \int f_n \, d\lambda = 1$. But

$$\lim_{n \to \infty} \int f_n \, d\lambda = \int 0 \, d\lambda = 0 < 1 = \lim_{n \to \infty} \int f_n \, d\lambda.$$ 

Hence strict inequality holds in Fatou’s lemma with $E = \mathbb{R}$.

75 Suppose that $f$ is a nonnegative $M$-measurable function and $\{E_n\}_{n=1}^\infty$ is a sequence of Lebesgue measurable sets with $E_1 \supset E_2 \supset \cdots$. Further suppose $\int_E f \, d\lambda < \infty$. Prove that

$$\int_{\bigcap_{n=1}^\infty E_n} f \, d\lambda = \lim_{n \to \infty} \int_{E_n} f \, d\lambda.$$ 

Let $E$ denote $\bigcap_{n=1}^\infty E_n$. We apply Theorem 3.19 with $f_n = f|_{E_n}$; then since $f$ is a nonnegative and $E_n \supset E_{n+1}$ for all $n$ we have $(f|_{E_n})(x) \geq (f|_{E_{n+1}})(x)$ for all $n$ and $x \in \mathbb{R}$. Moreover, since $(f|_{E_n})(x) \leq f(x)$ for all $x \in \mathbb{R}$ we have $\int f|_{E_n} \, d\lambda \leq \int f \, d\lambda < \infty$. Observe that $\lim_{n \to \infty} \chi_{E_n}(x) = \chi_E(x)$ for all $x \in \mathbb{R}$; hence, $f\chi_{E_n} \to f\chi_E$ pointwise. Therefore we have

$$\int_E f \, d\lambda = \int f\chi_E \, d\lambda = \lim_{n \to \infty} \int f\chi_{E_n} \, d\lambda$$

by Theorem 3.19 (applied at (⋆)).

86 Suppose that $f$ is Lebesgue integrable over $E$ and that $\{E_n\}_{n=1}^\infty$ is a sequence of Lebesgue measurable sets with $E_1 \subset E_2 \subset \cdots$ and $\bigcup_{n=1}^\infty E_n = E$. Prove that

$$\int_E f \, d\lambda = \lim_{n \to \infty} \int_{E_n} f \, d\lambda.$$ 

We apply the Dominated Convergence Theorem (Theorem 3.22 on p154) with $g = |f|\chi_E$ and $f_n = f\chi_{E_n}$ (note that these are Lebesgue measurable). Since $f$ is Lebesgue integrable over $E$, $|f|\chi_E$ is Lebesgue integrable. Since $E_n \subset E$ we have

$$|f_n| = |f|\chi_{E_n} \leq |f|\chi_E = g.$$ 

Moreover, since $\bigcup_{n=1}^\infty E_n = E$ and $E_n \subset E_{n+1}$, we have $f_n = f\chi_{E_n} \to f\chi_E$ pointwise. Hence

$$\int_E f \, d\lambda = \int f\chi_E \, d\lambda = \lim_{n \to \infty} \int f\chi_{E_n} \, d\lambda$$

by the Dominated Convergence Theorem (applied at (⋆)).
88 **Bounded convergence theorem (BCT):** Let $E \in \mathcal{M}$ with $\lambda(E) < \infty$ and \{f_n\}_{n=1}^{\infty} a sequence of Lebesgue measurable functions that converges pointwise on $E$ to a real-valued function. Further suppose that there is an $M \in \mathbb{R}$ such that $|f_n(x)| \leq M$ for $n \in \mathbb{N}, x \in E$. Show that

$$\int_E \lim_{n \to \infty} f_n \, d\lambda = \lim_{n \to \infty} \int_E f_n \, d\lambda.$$ 

We apply the version of the Dominated Convergence Theorem given in the preceding exercise (Ex. 3.87). Let $g$ be the constant function $M$; then $g$ is Lebesgue integrable over $E$ since

$$\int_E g \, d\lambda = \int M \chi_E \, d\lambda = M \lambda(E) < \infty.$$ 

Moreover we have $|f_n(x)| \leq g(x)$ for $n \in \mathbb{N}, x \in E$ and the sequence \{f_n\}_{n=1}^{\infty} converges pointwise on $E$ to a real-valued function. The desired result now follows by Ex. 3.87 above.

Alternatively, one can apply the Dominated Convergence Theorem (Theorem 3.22 on p.154) using $g = M \chi_E$, then since $g$ is integrable, $|f_n \chi_E| \leq g$ for all $n \in \mathbb{N}$ and the fact that $\lim_{n \to \infty} f_n \chi_E = \chi_E \lim_{n \to \infty} f_n$ the desired result follows. \hfill \Box

89 Construct an example where $|f_n| \leq M$ for all $n \in \mathbb{N}, f_n \to f$ pointwise but $\int f_n \, d\lambda \not\to \int f \, d\lambda$. Why doesn’t this contradict the BCT?

For each $n \in \mathbb{N}$ set $f_n = \chi_{[n,n+1]}$; then $|f_n| \leq 1$ for all $n \in \mathbb{N}$ and $f_n \to f$ where $f$ is the constant function $0$. But $\int f_n \, d\lambda = \int \chi_{[n,n+1]} \, d\lambda = 1$ for all $n \in \mathbb{N}$ and $\int f \, d\lambda = 0$; so $\int f_n \, d\lambda \not\to \int f \, d\lambda$. This does not contradict the Bounded Convergence Theorem (see Ex. 3.88) because we are using $E = \mathbb{R}$ and $\lambda(E) = \infty$; that is, a key hypothesis is not satisfied, namely $\lambda(E) < \infty$. \hfill \Box

92 Show that, if $f$ is Lebesgue integrable (over $\mathbb{R}$) and the improper Riemann integral exists, then

$$\int_{-\infty}^{\infty} f(x) \, d\lambda(x) = \int_{-\infty}^{\infty} f(x) \, dx.$$ 

Let $f$ be a measurable function which is Lebesgue integrable, that is $\int |f| \, d\lambda < \infty$, and suppose that the improper Riemann integral exists. Then both improper integrals $\int_{0}^{\infty} f(x) \, dx$ and $\int_{-\infty}^{0} f(x) \, dx$ converge, that is, both

$$\int_{0}^{\infty} f(x) \, dx = \lim_{t \to \infty} \int_{0}^{t} f(x) \, dx \quad \text{and} \quad \int_{-\infty}^{0} f(x) \, dx = \lim_{v \to -\infty} \int_{v}^{0} f(x) \, dx$$

exist and are finite. Moreover we have

$$\int_{-\infty}^{\infty} f(x) \, dx = \int_{0}^{\infty} f(x) \, dx + \int_{-\infty}^{0} f(x) \, dx.$$ 

Now set $E_n = [0, n]$ for $n \in \mathbb{N}$ and $E = [0, \infty)$, then $E_n \subset E_{n+1}$ and $E = \cup E_n$; then since $f$ is Lebesgue integrable we have $\int_{E} f \, d\lambda = \lim_{n \to \infty} \int_{E_n} f \, d\lambda$ by Ex. 3.86 (p.159). Since the improper Riemann integral exists, $f$ is Riemann integrable on $[0, n]$ for all $n$ and thus by Theorem 3.23 (p.157) we have $\int_{E_n} f \, d\lambda = \int_{0}^{n} f(x) \, dx$ for all $n$. It follows that

$$\int_{E} f \, d\lambda = \lim_{n \to \infty} \int_{E_n} f \, d\lambda = \lim_{n \to \infty} \int_{0}^{n} f(x) \, dx = \int_{0}^{\infty} f(x) \, dx$$

A similar argument shows that $\int_{F} f \, d\lambda = \int_{-\infty}^{0} f(x) \, dx$ where $F = (-\infty, 0]$. It follows that

$$\int_{-\infty}^{\infty} f(x) \, dx = \int_{0}^{\infty} f(x) \, dx + \int_{-\infty}^{0} f(x) \, dx = \int_{E} f \, d\lambda + \int_{F} f \, d\lambda.$$ 

Since $\lambda(E \cap F) = 0$ we have $\int_{E \cap F} f \, d\lambda = 0$. Hence, $\int f \, d\lambda = \int_{E} f \, d\lambda + \int_{F} f \, d\lambda$. Thus $\int f \, d\lambda = \int_{-\infty}^{\infty} f(x) \, dx$ as required. \hfill \Box

105 Show that the DCT (page 154) remains valid if convergence pointwise is replaced by convergence $\lambda$-ae. In other words, suppose that \{f_n\}_{n=1}^{\infty} is a sequence of Lebesgue measurable functions that converges $\lambda$-ae to a real-valued function. Further suppose that there is a nonnegative Lebesgue integrable function, $g$, such that $|f_n| \leq g$ for all $n \in \mathbb{N}$. Prove that

$$\int_{E} \lim_{n \to \infty} f_n \, d\lambda = \lim_{n \to \infty} \int_{E} f_n \, d\lambda.$$ 

for each $E \in \mathcal{M}$. Note: $\lim_{n \to \infty} f_n$ is not defined on all of $\mathbb{R}$ unless, of course, \{f_n\}_{n=1}^{\infty} converges everywhere.

We are given a sequence of Lebesgue measurable functions \{f_n\}_{n=1}^{\infty} that converges $\lambda$-ae; so there is a measurable
set \( D \) such that \( \lim_{n \to \infty} f_n(x) \) exists for all \( x \in D \) and \( \lambda(D^c) = 0 \). Set \( h_n := f_n \chi_D \) for \( n \in \mathbb{N} \). Then the sequence \( \{h_n\}_{n=1}^{\infty} \) converges pointwise; indeed we have

\[
\lim_{n \to \infty} h_n(x) = \begin{cases} 
\lim_{n \to \infty} f_n(x) & \text{if } x \in D \\
0 & \text{else.}
\end{cases}
\]

Since \( |f_n| \leq g \) for all \( n \in \mathbb{N} \) we have \( |h_n| \leq g \) for all \( n \in \mathbb{N} \) and so we may apply the Dominated Convergence Theorem (on p 154) to obtain

\[
\int_E \lim_{n \to \infty} h_n \, d\lambda = \lim_{n \to \infty} \int_E h_n \, d\lambda
\]

for each \( E \in \mathcal{M} \). Let \( n \in \mathbb{N} \) we have \( h_n = f_n \lambda \text{-ae} \) and so by Proposition 3.15 (on p 163), \( \int_E h_n \, d\lambda = \int_E f_n \, d\lambda \) for each \( E \in \mathcal{M} \). Since \( \lim_{n \to \infty} h_n(x) = \lim_{n \to \infty} f_n(x) \) for all \( x \in D \) (and \( \lambda(D^c) = 0 \)) we have by Definition 3.18

\[
\int_E \lim_{n \to \infty} h_n \, d\lambda = \int_E \lim_{n \to \infty} f_n \, d\lambda
\]

for each \( E \in \mathcal{M} \) (note that \( \lim_{n \to \infty} f_n \) may not be defined everywhere). Hence we have

\[
\int_E \lim_{n \to \infty} f_n \, d\lambda = \int_E \lim_{n \to \infty} h_n \, d\lambda = \lim_{n \to \infty} \int_E h_n \, d\lambda = \lim_{n \to \infty} \int_E f_n \, d\lambda
\]

for each \( E \in \mathcal{M} \) and the result follows. \( \square \)

108 Let \( f \) and \( g \) be \( \mathcal{M} \)-measurable functions with \( \int |f - g| \, d\lambda = 0 \). Prove that \( f = g \lambda \text{-ae} \).

We proved the following Proposition in class: If \( h \) is a nonnegative \( \mathcal{M} \)-measurable function such that \( \int h \, d\lambda = 0 \), then \( h = 0 \lambda \text{-ae} \).

Since \( h = |f - g| \) is a nonnegative \( \mathcal{M} \)-measurable function with \( \int h \, d\lambda = 0 \), it follows that \( |f - g| = 0 \lambda \text{-ae} \). That is, there is a measurable set \( E \) such that \( \lambda(E) = 0 \) and \( |f(x) - g(x)| = 0 \) for all \( x \in E^c \). Hence for \( x \in E^c \), we have \( f(x) - g(x) = 0 \) and therefore \( f(x) = g(x) \). Therefore \( f = g \lambda \text{-ae} \). \( \square \)

109 Show that, if \( f \) is Lebesgue integrable and \( \int_E f \, d\lambda = 0 \) for each \( E \in \mathcal{M} \), then \( f = 0 \lambda \text{-ae} \).

Let \( f \) be as above. Set \( E = f^{-1}((0, \infty)) \) and \( F = f^{-1}((\infty, 0)) \). It suffices to show that \( \lambda(E) = \lambda(F) = 0 \). Note that \( E = \bigcup_{n=1}^{\infty} E_n \) where \( E_n = f^{-1}([\frac{1}{n}, \infty)) \). Let \( n \in \mathbb{N} \) be given. Since \( f(x) \geq \frac{1}{n} \) for all \( x \in E_n \) we have \( f \chi_{E_n} \geq \frac{1}{n} \chi_{E_n} \). Hence, we have (by Proposition 3.10(a) on p 136)

\[
\int_{E_n} f \, d\lambda = \int f \chi_{E_n} \, d\lambda \geq \int \frac{1}{n} \chi_{E_n} \, d\lambda = \frac{1}{n} \lambda(E_n).
\]

But since we are given that \( \int_E f \, d\lambda = 0 \) for each \( E \in \mathcal{M} \), it follows that for each \( n \in \mathbb{N} \) we have

\[
0 \leq \lambda(E_n) \leq n \int_{E_n} f \, d\lambda = 0
\]

and so \( \lambda(E_n) = 0 \). Hence, since \( E_n \subseteq E_{n+1} \) for all \( n \) and \( E = \bigcup_{n=1}^{\infty} E_n \), we have \( \lambda(E) = \lim_{n \to \infty} \lambda(E_n) = 0 \) (by Theorem 4.1(d) on p 170). A similar argument shows \( \lambda(F) = 0 \). Hence, \( f = 0 \lambda \text{-ae} \). \( \square \)