2.3 For \( x, y \in \mathbb{R} \), we define the **maximum** of \( x \) and \( y \) to be the larger of those two numbers. We denote the maximum by \( \max\{x, y\} \) or \( x \vee y \). Thus,

\[
\max\{x, y\} = x \vee y = \begin{cases} 
  x, & \text{if } x \geq y; \\
  y, & \text{if } x < y.
\end{cases}
\]

Similarly, we define the **minimum** of \( x \) and \( y \) to be the smaller of those two numbers. We denote the minimum by \( \min\{x, y\} \) or \( x \wedge y \). Thus,

\[
\min\{x, y\} = x \wedge y = \begin{cases} 
  y, & \text{if } x \geq y; \\
  x, & \text{if } x < y.
\end{cases}
\]

Let \( x, y \in \mathbb{R} \). Referring to Exercise 2.2, prove each of the following facts.

a. \( |x| = x \vee -x \)

There are two cases to consider: either \( x \geq 0 \) or \( x < 0 \). If \( x \geq 0 \), then \( x \geq -x \) and so \( x \vee -x = x = |x| \). If \( x < 0 \), then \( -x \geq x \) and so \( x \vee -x = -x = |x| \). Hence, \( |x| = x \vee -x \) for all \( x \in \mathbb{R} \).

b. \( x \vee y = \frac{1}{2}(x + y + |x - y|) \)

There are two cases to consider: either \( x \geq y \) or \( x < y \). If \( x \geq y \) then \( |x - y| = x - y \) and hence

\[
\frac{1}{2}(x + y + |x - y|) = \frac{1}{2}(x + y + (x - y)) = x = x \vee y.
\]

If \( x < y \) then \( |x - y| = y - x \) and hence

\[
\frac{1}{2}(x + y + |x - y|) = \frac{1}{2}(x + y + (y - x)) = y = x \vee y.
\]

The result follows.

c. \( x \wedge y = \frac{1}{2}(x + y - |x - y|) \)

Clearly \( x \wedge y + x \vee y = x + y \). Hence, by part (b) we have

\[
x \wedge y = x + y - x \vee y = x + y - \frac{1}{2}(x + y + |x - y|) = \frac{1}{2}(x + y - |x - y|)
\]

as required.

2.4 Suppose that \( A \) is bounded below. Prove that \( A \) has a greatest lower bound and that, in fact,

\[
\inf A = -\sup\{-x : x \in A\}.
\]

Suppose that \( A \) is a nonempty set of real numbers which is bounded below, then \( \sup\{-x : x \in A\} \) is bounded above and therefore has a least upper bound by the Completeness Axiom. Set \( l = -\sup\{-x : x \in A\} \). To show that \( l = \inf A \) we must show that \( l \) is a lower bound for \( A \) and that if \( l' > l \) then \( l' \) is not a lower bound for \( A \), that is, there is an element \( x \in A \) such that \( x < l' \). Let \( x \in A \), then since \( -l = \sup\{-x : x \in A\} \), we have \( -x \leq -l \) and thus \( l \leq x \); hence, \( l \) is a lower bound for \( A \). Let \( l' \) be given so that \( l' > l \). Then since \( -l' < -l = \sup\{-x : x \in A\} \), \( -l' \) is not an upper bound for \( \{-x : x \in A\} \). Hence, there is \( x \in A \) so that \( -l' < -x \). Therefore, \( x < l' \) and \( l' \) is not a lower bound for \( A \). Hence \( l = \inf A \).

2.8 Prove that any (nondegenerate) interval of real numbers contains infinitely many rational numbers.

Let \( I \) be a nondegenerate interval of real numbers; then there are two real numbers \( a, b \in I \) with \( a < b \) such that \((a, b) \subseteq I\). By Proposition 2.4 there is a rational number \( r_0 \in (a, b) \) and a second rational number \( r_1 \in (r_0, b) \). We construct a decreasing sequence in \( I \) recursively starting with \( r_1 \) by setting \( r_{n+1} = (r_n + r_0)/2 \). We have \( r_n \in (r_0, b) \subseteq I \) for all \( n \in \mathbb{N} \) and the sequence is decreasing so the terms are distinct. Therefore there are infinitely many rational numbers in \( I \). Note that we have used the fact that if \( c, d \in \mathbb{Q} \) then \((c + d)/2 \in \mathbb{Q} \) and \( c < (c + d)/2 < d \).

2.16a Prove Proposition 2.6 on page 45.

Any monotone sequence of real numbers converges in \( \mathbb{R}^* \). In fact, we have the following: If \( \{x_n\}_{n=1}^\infty \) is nondecreasing, then

\[
\lim_{n \to \infty} x_n = \sup\{x_n : n \in \mathbb{N}\}.
\]

In particular, the limit exists and is finite if \( \{x_n\}_{n=1}^\infty \) is bounded above and is \( \infty \) otherwise.

Let \( \{x_n\}_{n=1}^\infty \) be a nondecreasing sequence of real numbers. First suppose the sequence is bounded above. Then by the completeness axiom \( x = \sup\{x_n : n \in \mathbb{N}\} \) exists and is finite. We now show that \( x = \lim_{n \to \infty} x_n \). Given \( \varepsilon > 0 \), since \( x - \varepsilon \) is not an upper bound for the set \( \{x_n : n \in \mathbb{N}\} \) (\( x \) is the least upper bound) there is \( N \in \mathbb{N} \) such that \( x - \varepsilon < x_N \leq x \). Since \( \{x_n\}_{n=1}^\infty \) is nondecreasing we have \( x - \varepsilon < x_N \leq x_n \leq x \) for all \( n \geq N \). It follows that \( |x - x_n| < \varepsilon \) for all \( n \geq N \). Hence, \( x = \lim_{n \to \infty} x_n \); moreover, the limit exists and is finite. Suppose that \( \{x_n : n \in \mathbb{N}\} \) is not bounded above, then \( \sup\{x_n : n \in \mathbb{N}\} = \infty \). We now show that \( \lim_{n \to \infty} x_n = \infty \). Let \( M \in \mathbb{R} \) be given; then since \( \{x_n : n \in \mathbb{N}\} \) is not bounded above, \( M \) is not an upper bound and therefore there is \( N \in \mathbb{N} \) such that \( x_N > M \).
Since \( \{x_n\}_{n=1}^\infty \) is nondecreasing we have \( x_n \geq x_N > M \) for all \( n \geq N \). Hence, \( \lim_{n \to \infty} x_n = \infty \). In either case, \( \lim_{n \to \infty} x_n = \sup\{x_n : n \in \mathbb{N}\} \).

2.28a Let \( \{x_n\}_{n=1}^\infty \) and \( \{y_n\}_{n=1}^\infty \) and assume that \( \lim_{n \to \infty} y_n \) exists and is finite. Prove that

\[
\limsup(x_n + y_n) = \limsup x_n + \lim y_n.
\]

Let \( x = \limsup x_n \) and \( y = \lim y_n \). We first assume that \( x \) is a real number and use Proposition 2.8(a) to verify that \( \limsup(x_n + y_n) = x + y \). Let \( \varepsilon > 0 \) be given. Since \( x = \limsup x_n \) there is an \( N_1 \in \mathbb{N} \) such that \( x_n \leq x + \varepsilon/2 \) for all \( n \geq N_1 \). Since \( y_n \to y \), there is \( N_2 \in \mathbb{N} \) such that \( y - y/2 < y_n < y + \varepsilon/2 \) for all \( n \geq N_2 \). Then for all \( n \geq N = \max\{N_1, N_2\} \) both conditions hold and we have \( x + y_n \leq x + y + \varepsilon \). Now, let \( n \in \mathbb{N} \) be given, then since \( x = \limsup x_n \) there is \( m \geq \max(n, N_2) \) such that \( x_m > x - \varepsilon/2 \) and since \( m \geq N_2 \), \( y_m > y - \varepsilon/2 \). Hence, \( x_m + y_m > x + y - \varepsilon \) for some \( m \geq n \). Hence, \( \limsup(x_n + y_n) = x + y \).

Now suppose that \( \limsup x_n = \infty \), then we use Proposition 2.8(b) to show that

\[
\limsup(x_n + y_n) = \limsup x_n + \lim y_n = \infty + \infty = \infty.
\]

Since \( y_n \to y \) there is a \( K \in \mathbb{N} \) such that \( y_n > y - 1 \) for all \( n \geq K \). Now let \( M \in \mathbb{R} \) and \( N \in \mathbb{N} \) be given. Since \( \limsup x_n = \infty \), there is \( n \geq \max\{N, K\} \) such that \( x_n > M - y + 1 \). Since \( n \geq K \) we have \( y_n > y - 1 \). Hence, \( x_n + y_n > M - y + 1 + y - 1 = M \).

Hence, \( \limsup(x_n + y_n) = \infty \) as required.

Note that by Proposition 2.8(c) we have \( \limsup x_n = -\infty \) iff \( \lim_{n \to \infty} x_n = -\infty \). Now suppose that \( \lim sup x_n = -\infty \) or equivalently that \( \lim_{n \to \infty} x_n = -\infty \). Then to show that

\[
\limsup(x_n + y_n) = \limsup x_n + \lim y_n = -\infty + y = -\infty.
\]

it suffices by Proposition 2.8(c) to show that \( \lim_{n \to \infty} x_n + y_n = -\infty \). Now let \( M \in \mathbb{R} \). Since \( y_n \to y \) there is a \( N_1 \in \mathbb{N} \) such that \( y_n < y + 1 \) for all \( n \geq N_1 \). Since \( x_n \to -\infty \), there is a \( N_2 \in \mathbb{N} \) such that \( x_n < M - y - 1 \) for all \( n \geq N_1 \). Then for all \( n \geq N = \max\{N_1, N_2\} \) we have

\[
x_n + y_n < M - y - 1 + y + 1 = M.
\]

Hence, \( \lim_{n \to \infty} x_n + y_n = -\infty \).

2.36 In this exercise, we will discuss infinite series. Let \( \{x_n\}_{n=1}^\infty \) be a sequence of real numbers. The sequence \( \{s_n\}_{n=1}^\infty \) defined by \( s_n = \sum_{k=1}^n x_k , n \in \mathbb{N} \), is called the sequence of partial sums of \( \{x_n\}_{n=1}^\infty \). If the sequence \( \{s_n\}_{n=1}^\infty \) converges to a real number, say, \( s \), then we say that \( \{x_n\}_{n=1}^\infty \) is summable to \( s \) or that the infinite series \( \sum_{n=1}^\infty x_n \) converges to \( s \), and we write \( s = \sum_{n=1}^\infty x_n \). We also say that \( s \) is the sum of the infinite series. If the sequence \( \{s_n\}_{n=1}^\infty \) does not converge to a real number, then we say that \( \{x_n\}_{n=1}^\infty \) is not summable or that the infinite series \( \sum_{n=1}^\infty x_n \) diverges. For brevity we often write \( \sum x_n \) in place of \( \sum_{n=1}^\infty x_n \).

a) Prove that if \( x_n \geq 0 \) for each \( n \in \mathbb{N} \), then either \( \lim_{n \to \infty} s_n = \infty \) or \( \sum x_n \) converges.

Suppose \( x_n \geq 0 \) for each \( n \in \mathbb{N} \). Then the sequence \( \{s_n\}_{n=1}^\infty \) given by \( s_n = \sum_{k=1}^n x_k \) is a nondecreasing sequence. Hence, by Proposition 2.6 either the sequence converges to \( \infty \) if \( \{s_n\}_{n=1}^\infty \) is not bounded above or it is bounded above and thus has a finite limit, in which case \( \sum x_n \) converges.

b) Show that if \( \sum x_n \) converges, then \( \lim_{n \to \infty} x_n = 0 \)

Suppose \( \sum x_n \) converges. Then the sequences \( \{s_n\}_{n=1}^\infty \) and \( \{s_{n+1}\}_{n=1}^\infty \) share the same finite limit \( s \), hence,

\[
\lim_{n \to \infty} x_{n+1} = \lim_{n \to \infty} s_{n+1} - s_n = \lim_{n \to \infty} s_{n+1} - \lim_{n \to \infty} s_n = s - s = 0.
\]

It follows that \( \lim_{n \to \infty} x_n = 0 \).

c) Show that if \( \sum x_n \) converges, then \( \lim_{n \to \infty} \sum_{k=n}^\infty x_k = 0 \).

Suppose that \( \sum_{n=1}^\infty x_n \) converges, say, \( s = \sum_{n=1}^\infty x_n \). Then \( \sum_{k=n}^\infty x_k \) also converges and if \( n \geq 2 \) we have \( s = \sum_{k=n}^\infty x_k = s - s_{n-1} \). Hence, \( \lim_{n \to \infty} \sum_{k=n}^\infty x_k = \lim_{n \to \infty} s - s_{n-1} = 0 \).

d) Prove that if \( \sum |x_n| \) converges, then so does \( \sum x_n \). Hint: Use the Cauchy criterion.

Suppose that \( \sum |x_n| \) converges, then the sequence of partial sums \( \{t_n\}_{n=1}^\infty \) given by \( t_n = \sum_{k=1}^n |x_k| \) converges and therefore is a Cauchy sequence (by the Cauchy Criterion: Theorem 2.1). To show that \( \sum x_n \) converges it suffices to show that its sequence of partial sums \( \{s_n\}_{n=1}^\infty \) given by \( s_n = \sum_{k=1}^n x_k \) is a Cauchy sequence. Let \( \varepsilon > 0 \) be given. Then since \( \{t_n\}_{n=1}^\infty \) is Cauchy, there is \( N \in \mathbb{N} \) such that \( |t_m - t_n| < \varepsilon \) for all \( m, n \geq N \). Now let \( m, n \geq N \), we show that \( |s_m - s_n| < \varepsilon \). If \( m = n \) there is nothing to prove; by switching \( m \) and \( n \) if necessary we may assume that \( n < m \). By the triangle inequality we have

\[
|s_m - s_n| = \sum_{k=1}^n |x_k| - \sum_{j=1}^{n-1} |x_j| = \sum_{k=n+1}^m |x_k| \leq \sum_{k=n+1}^m |t_k - t_{k-1}| < \varepsilon.
\]

Hence, \( \{s_n\}_{n=1}^\infty \) is Cauchy and the series converges.