Math 713 - Review Sheet for the final exam  
Tuesday, 5 December 2006

The Final Exam is on Monday, 18 Dec, noon-2. It will cover up through §4.3. Study the review sheet given for Test 2 in addition to this one. You will have to choose approximately 80% of the questions on the exam and solve them correctly for full points.

(1) Show that a nonempty set $A$ is countable iff there is a surjective function $f : \mathbb{N} \to A$.
(2) Let $\Omega$ be a nonempty countable set and suppose that $\mathcal{A}$ is a $\sigma$-algebra of subsets of $\Omega$ such that for any pair of distinct elements $x, y \in \Omega$ there is an $A \in \mathcal{A}$ such that $x \in A$ but $y \notin A$. Show that $\mathcal{A} = \mathcal{P}(\Omega)$.
(3) Give an example (if possible) of a $\sigma$-algebra which is not an algebra.
(4) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^3$. Use the definition of continuity to prove that $f$ is continuous.
(5) Let $f : X \to Y$ be a function and suppose that $A$ and $B$ are subsets of $Y$. Show that $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$, $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$, and $f^{-1}(A^c) = f^{-1}(A)^c$.
(6) Suppose that $\{x_n\}_{n=1}^{\infty}$ is a sequence of real numbers which converges to $x_0$ and that $f : \mathbb{R} \to \mathbb{R}$ is continuous at $x_0$. Show that $f(x_n) \to f(x_0)$ as $n \to \infty$.
(7) Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of real numbers show that $\limsup x_n = \infty$ iff $\sup_n x_n = \infty$ and that $\liminf x_n = \infty$ iff $\lim_{n \to \infty} x_n = \infty$.
(8) Let $f$ and $g$ be real valued continuous functions defined on $\mathbb{R}$ show that $f \vee g$ is continuous.
(9) Suppose that $f : X \to Y$ is a function and $\mathcal{A}$ is a $\sigma$-algebra of sets on $Y$. Show that the following collection of subsets of $X$ is also a $\sigma$-algebra

$$\{f^{-1}(A) : A \in \mathcal{A}\}.$$  
(10) Suppose that $f : A \to B$ is one-to-one. Show that $A$ is countable if $B$ is. Does the converse hold?
(11) Suppose that $f : A \to B$ is a surjective function. Show that there is a function $g : B \to A$ such that $f(g(b)) = b$ for all $b \in B$. Must $g$ be one-to-one?
(12) Let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be two sequences of real numbers. Show that $\lim sup(x_n + y_n) \leq \lim sup x_n + \lim sup y_n$ if the right hand side makes sense.
(13) Let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be two sequences of real numbers and suppose that $x_n \leq y_n$ for all $n \in \mathbb{N}$. Show that $\lim inf x_n \leq \lim inf y_n$.
(14) Show that every bounded sequence of real numbers has a convergent subsequence.
(15) Define a closed set. Prove that a finite union of closed sets is closed.
(16) State the Monotone Convergence Theorem and prove two interesting corollaries.
(17) Let $E = [0, 1]$ and define the functions $f$ and $f_n$ for each $n \in \mathbb{N}$ by

$$f(x) = \begin{cases} \frac{1}{x^2} & \text{for } 0 \leq x < 1 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad f_n(x) = \begin{cases} 1 + x + \cdots + x^n & \text{for } 0 \leq x < 1 \\ 0 & \text{otherwise} \end{cases}.$$  
Show that

$$\int_E f \, d\lambda = \lim_{n \to \infty} \int_E f_n \, d\lambda.$$  
(18) State and prove Fatou's Lemma for a sequence of nonnegative real-valued $\mathcal{M}$-measurable functions defined on $\mathbb{R}$.
(19) State the Dominated Convergence Theorem.
(20) Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \sum_{n=1}^{\infty} 2^{-n} \cos nx.$$  
Show that $f$ is well-defined by verifying that the series converges for all $x \in \mathbb{R}$. Show that $f$ is a Borel measurable function and that $f$ is integrable on $E = [0, \pi]$. Evaluate $\int_E f \, d\lambda$.
(21) Suppose that the function $f : \mathbb{R} \to \mathbb{R}$ is continuous and nonnegative. Show that $f$ is $\mathcal{M}$-measurable and that $\int_a^b f(x) \, dx = \int_a^b f \, d\lambda$ where $E = [a, b]$.
(22) Definition 4.1 on page 168, Theorem 4.1 on page 170; Definition 4.4, Propositions 4.1 and 4.2 on pages 175-6.
(23) Let $(\Omega, \mathcal{A})$ be a measurable space. Define what it means for a function $f : \Omega \to \mathbb{R}$ to be simple. Show that the collection of all simple functions forms an algebra.
(24) Let $(\Omega, \mathcal{A})$ be a measurable space and let $f$ be a real-valued function defined on $\Omega$. Show that $f$ is $\mathcal{A}$-measurable iff $f^{-1}(B) \in \mathcal{A}$ for each Borel set $B$.
(25) Carefully state the Monotone Convergence Theorem for extended real valued functions on a measure space and prove two interesting corollaries.
(26) Let $(\Omega, \mathcal{A})$ be a measurable space. Suppose that $f$ and $g$ are extended real-valued $\mathcal{A}$-measurable functions defined on $\Omega$. Prove that $\{x : f(x) = g(x)\} \in \mathcal{A}$ is $\mathcal{A}$-measurable.
(27) Let $(\Omega, \mathcal{A})$ be a measurable space and let $\{f_n\}_{n=1}^{\infty}$ be a sequence of real-valued $\mathcal{A}$-measurable functions. Show that $E = \{x \in \Omega : \lim_{n \to \infty} f_n(x) \text{ exists} \} \in \mathcal{A}$; further, show that the function $f : \Omega \to \mathbb{R}$ given below is $\mathcal{A}$-measurable

$$f(x) = \begin{cases} \lim_{n \to \infty} f_n(x) & \text{if } x \in E, \\ 0 & \text{else.} \end{cases}$$  
(28) Consider the measure space $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu)$ where $\mu$ is the counting measure and let $f : \mathbb{N} \to \mathbb{R}$ be a function. Show that $f$ is $\mathcal{P}(\mathbb{N})$-measurable. Suppose that $f$ is nonnegative and use the definition to show that $\int f \, d\mu = \sum_{n=1}^{\infty} f(n)$.
Let \( \{f_n\}_{n=1}^\infty \) and \( \{g_n\}_{n=1}^\infty \) are two sequences of real-valued \( \mathcal{M} \)-measurable functions on \( \mathbb{R} \) such that \( f_n = g_n \) \( \lambda \)-ae for every \( n \in \mathbb{N} \). Show that \( \{f_n\}_{n=1}^\infty \) converges \( \lambda \)-ae iff \( \{g_n\}_{n=1}^\infty \) converges \( \lambda \)-ae. In this case what can one say about their respective limits?

Exercise 3.102.

Find if possible an example of a real-valued \( \mathcal{M} \)-measurable function \( f \) which is not Lebesgue integrable on \([0, \infty)\) such that the improper Riemann integral \( \int_0^\infty f(x) \, dx \) converges. Does your answer change if \( f \) is required to be nonnegative?

Show that the function \( f : \mathbb{R} \to \mathbb{R} \) given by \( f(x) = e^{-x} \sin(x^2) \) is integrable on \([0, \infty)\).

Let \( f : \mathbb{R} \to \mathbb{R} \) be the function given by
\[
  f(x) = \begin{cases} 
    x^{-1/3} & \text{if } x \neq 0 \\
    0 & \text{else} 
  \end{cases}.
\]

Show that \( f \) is integrable on \( E_a = [-a, a] \) for all \( a > 0 \). Is \( f \) integrable on \( \mathbb{R} \)?

Let \( E_t = [0, t] \) for \( 0 < t < 1 \) and define the functions \( f \) and \( f_n \) for \( n \in \mathbb{N} \) by
\[
  f(x) = \begin{cases} 
    \frac{1}{1+x} & \text{for } 0 \leq x < 1 \\
    0 & \text{otherwise}
  \end{cases}
\]
\[
  f_n(x) = \begin{cases} 
    \sum_{k=0}^{\infty} (-1)^k x^k & \text{for } 0 \leq x < 1 \\
    0 & \text{otherwise}
  \end{cases}
\]

Show that
\[
  \int_{E_t} f \, d\lambda = \lim_{n \to \infty} \int_{E_t} f_n \, d\lambda.
\]