(4.3) Consider the experiment of selecting a number at random from the closed interval $[-1, 1]$.

(a) Construct an appropriate probability space for this experiment.

**Solution:** Set $\Omega := [-1, 1]$ and $\mathcal{A} := \mathcal{M}_2$. Thus $\mathcal{A}$ is a $\sigma$-algebra. We let $P : \mathcal{A} \to [0, 1]$ be defined by $P(E) = \frac{\lambda(E)}{2}$. Note that

(a) for any $E \in \mathcal{A}$ we have $P(E) = \frac{\lambda(E)}{2} \geq 0$ since $\lambda(E) \geq 0$.

(b) for any $E \in \mathcal{A}$ we have $P(\emptyset) = \frac{\lambda(\emptyset)}{2} = \frac{0}{2} = 0$ since $\lambda(\emptyset) = 0$.

(c) for $\{E_n\} \subset \mathcal{A}$ where $E_i \cap E_j = \emptyset$ for $i \neq j$ we have

$$P \left( \bigcup_n E_n \right) = \frac{\lambda \left( \bigcup_n E_n \right)}{2} = \sum_n \frac{\lambda(E_n)}{2} = \sum_n \frac{\lambda(E_n)}{2} = \sum_n P(E_n)$$

Thus $P$ is a measure, and furthermore

$$P(\Omega) = P([-1, 1]) = \frac{\lambda([-1, 1])}{2} = \frac{2}{2} = 1$$

So $P$ is a probability measure. Thus our probability space is $(\Omega, \mathcal{A}, P)$.

(b) Determine the probability that the number selected exceeds .5.

**Solution:** We have

$$P(E) = \frac{\lambda((.5, 1])}{2} = \frac{.5}{2} = \frac{1}{4}$$

1. Determine the probability that the number selected is rational.

**Solution:** We set $\Omega := [-1, 1] \cap \mathbb{Q}$. We have that

$$P(E) = P([-1, 1] \cap \mathbb{Q}) = \frac{\lambda([-1, 1] \cap \mathbb{Q})}{2} = \frac{0}{2} = 0,$$

where (1) was obtained from the fact that $\lambda(\mathbb{Q}) = 0$, thus $\lambda(E) = 0$ since $E \subset \mathbb{Q}$.

(4.7) Suppose that a balanced coin is tossed three times.

(a) Construct a probability space for this experiment in which each possible outcome is equally likely.

**Solution:** Set $\Omega := \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$. (Each element is a sequence of H’s and T’s, where an H in the first position denotes landing heads on the first coin toss and so on.) We set $\mathcal{A} := \mathcal{P}(\Omega)$. Thus $\mathcal{A}$ is a $\sigma$-algebra. We let $P : \mathcal{A} \to [0, 1]$ be defined by $P(E) = \frac{|E|}{8}$. And we see that

(a) for any $E \in \mathcal{A}$ we have $P(E) = \frac{|E|}{8} \geq 0$ since $|E| \geq 0$.

(b) for any $E \in \mathcal{A}$ we have $P(\emptyset) = \frac{|\emptyset|}{8} = \frac{0}{8} = 0$.

(c) for $\{E_n\} \subset \mathcal{A}$ where $E_i \cap E_j = \emptyset$ for $i \neq j$ we have

$$P \left( \bigcup_n E_n \right) = \frac{\sum_n |E_n|}{8} = \sum_n \frac{|E_n|}{8} = \sum_n P(E_n)$$

Thus $P$ is a measure, and furthermore

$$P(\Omega) = \frac{|\Omega|}{8} = \frac{8}{8} = 1$$

So $P$ is a probability measure. Thus our probability space is $(\Omega, \mathcal{A}, P)$.

(b) Determine the probability of obtaining exactly two heads.

**Solution:** Let $E$ be the event that one lands exactly two heads. Thus $E := \{HHT, HTH, THH\}$. And we see that

$$P(E) = \frac{|E|}{8} = \frac{3}{8}$$

(c) Express the probability measure, $P$, as a finite linear combination of Dirac measures.

**Solution:** With $x_0 = \Omega$ as in part (a) we have

$$\delta_{x_0} = \begin{cases} 1, & \text{if } x_0 \in E \\ 0, & \text{else} \end{cases}$$

And so $P$ can be written as

$$P = \frac{1}{8} \delta_{HHH} + \frac{1}{8} \delta_{HHT} + \frac{1}{8} \delta_{HTH} + \frac{1}{8} \delta_{THH} + \frac{1}{8} \delta_{THT} + \frac{1}{8} \delta_{TTH} + \frac{1}{8} \delta_{TTT} = \frac{1}{8} \sum_{i \in \Omega} \delta_i$$
(4.9) Suppose that two balanced dice are thrown.

(a) Construct a probability space for this experiment in which each possible outcome is equally likely.

Solution: Let $\Omega$ be the set of ordered pairs listed in the diamond below. Each ordered pair denotes the value of the roll of dice one and the value of the roll of dice two. Set $A := P(\Omega)$.

<table>
<thead>
<tr>
<th>Dice Combinations</th>
<th>Sum</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1, 1)</td>
<td>2</td>
</tr>
<tr>
<td>(1, 2)</td>
<td>3</td>
</tr>
<tr>
<td>(2, 1)</td>
<td>3</td>
</tr>
<tr>
<td>(2, 2)</td>
<td>4</td>
</tr>
<tr>
<td>(3, 1)</td>
<td>4</td>
</tr>
<tr>
<td>(3, 2)</td>
<td>5</td>
</tr>
<tr>
<td>(4, 1)</td>
<td>5</td>
</tr>
<tr>
<td>(4, 2)</td>
<td>6</td>
</tr>
<tr>
<td>(5, 1)</td>
<td>6</td>
</tr>
<tr>
<td>(5, 2)</td>
<td>7</td>
</tr>
<tr>
<td>(6, 1)</td>
<td>7</td>
</tr>
<tr>
<td>(6, 2)</td>
<td>8</td>
</tr>
<tr>
<td>(6, 3)</td>
<td>9</td>
</tr>
<tr>
<td>(6, 4)</td>
<td>10</td>
</tr>
<tr>
<td>(6, 5)</td>
<td>11</td>
</tr>
<tr>
<td>(6, 6)</td>
<td>12</td>
</tr>
</tbody>
</table>

Thus $A$ is a $\sigma$-algebra. Now let $P : A \to [0, 1]$ be defined by $P(E) = \frac{|E|}{36}$. We have

(a) for any $E \in A$ we have $P(E) = \frac{|E|}{36} \geq 0$ since $|E| \geq 0$.

(b) for any $E \in A$ we have $P(\emptyset) = \frac{|\emptyset|}{36} = \frac{0}{36} = 0$.

(c) for $\{E_n\} \subset A$ where $E_i \cap E_j = \emptyset$ for $i \neq j$ we have

$$P\left( \bigcup_n E_n \right) = \frac{\left| \bigcup_n E_n \right|}{36} = \sum_n \frac{|E_n|}{36} = \sum_n \frac{|E_n|}{36} = \sum_n P(E_n)$$

and so $P$ is a measure and furtheremore we have that

$$P(\Omega) = \frac{|\Omega|}{36} = \frac{36}{36} = 1.$$ 

So $P$ is a probability measure. Thus our probability space is $(\Omega, A, P)$.

(b) Use part (a) to determine the probability that the sum of the dice is seven or eleven.

Solution: Let $E$ be the even that the sum of the dice is seven or eleven. Thus $E = \{(6, 1), (5, 2), (4, 3), (3, 4), (2, 5), (1, 6), (6, 5), (5, 6)\}$ and so

$$P(E) = \frac{|E|}{36} = \frac{8}{36}$$

(c) Construct a probability space for this experiment in which each the outcomes consist of the possible sums of the two dice.

Solution: Set $\Omega := \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ and set $A := P(\Omega)$. Thus $A$ is a $\sigma$-algebra. We let $P : A \to [0, 1]$ be defined by $P(E) = \frac{\sum_{\omega \in E} \sigma_{\omega}}{36}$ where

$$\sigma_{\omega} = |\{(x_1, x_2), 1 \leq x_1, x_2 \leq 6 \mid x_1 + x_2 = \omega\}|.$$ 

Clearly, we have

(a) for any $E \in A$ we have $P(E) = \frac{\sum_{\omega \in E} \sigma_{\omega}}{36} \geq 0$ since $\sum_{\omega \in E} \sigma_{\omega} \geq 0$ ($\sigma_{\omega} \geq 0$).

(b) for any $E \in A$ we have $P(\emptyset) = \frac{0}{36} = 0$.

(c) for $\{E_n\} \subset A$ where $E_i \cap E_j = \emptyset$ for $i \neq j$ we have

$$P\left( \bigcup_n E_n \right) = \frac{\sum_{\omega \in \bigcup_n E_n} \sigma_{\omega}}{36} = \frac{\sum_{E \in \bigcup_n E_n} \sum_{\omega \in E} \sigma_{\omega}}{36} = \sum_n \frac{\sum_{\omega \in E_n} \sigma_{\omega}}{36} = \sum_n P(E_n)$$

and so $P$ is a measure and furtheremore we have that

$$P(\Omega) = \frac{\sum_{\omega \in \Omega} \sigma_{\omega}}{36} = \frac{\sigma_2 + \cdots + \sigma_{12}}{36} = \frac{36}{36} = 1.$$ 

So $P$ is a probability measure. Thus our probability space is $(\Omega, A, P)$.

(d) Use part (c) to determine the probability that the sum of the dice is seven or eleven.

Solution: Let $E$ be the even that the sum of the dice is seven or eleven. Thus $E = \{7, 11\}$. Thus

$$P(E) = \frac{\sum_{\omega \in E} \sigma_{\omega}}{36} = \frac{\sigma_7 + \sigma_{11}}{36} = \frac{6 + 2}{36} = \frac{8}{36}$$

where (1) can be obtained from the looking at the diamond in Figure 1 at the top of page 2 of this assigment.