You will have to choose approximately 80% of the questions on the exam and solve them correctly for full points. This will allow you some flexibility regarding what material to concentrate more on. The exam will be based on Sections: 3.1-3.6 (up to and including Theorem 3.17 only), and probability measure questions from 4.1.

You should be able to state the definitions: 3.1, 2, 6, 8, 9, 11, 13, 14.

Practice the homework problems:

Section 3.1: 3.1, 3.2, 3.3, 3.5, 3.6, 3.11
Section 3.2: 3.15-19
Section 3.3: 3.23-3.26
Section 3.4: 3.30-3.38, 3.44, 3.45, 3.47a, 3.50a, 3.51
Section 3.5: 3.53, 3.56a, 3.62
Section 3.6: 3.71
Section 4.1: 4.3, 4.7, 4.9

In addition, try the following:

1. Define what it means for a function to be simple. Show that the collection of all simple functions forms an algebra.
2. Let \( A \subset \mathbb{R} \) be given. Carefully define what is meant by the outer measure \( \lambda^*(A) \). Suppose that \( B \) is another subset of \( \mathbb{R} \); use the definition to show that \( \lambda^*(A \cup B) \leq \lambda^*(A) + \lambda^*(B) \).
3. Show that if \( f : \mathbb{R} \to \mathbb{R} \) is Borel measurable if \( f^{-1}(r, \infty) \in \mathcal{B} \) for all \( r \in \mathbb{Q} \). (Definition: A function \( f : \mathbb{R} \to \mathbb{R} \) is Borel measurable if for each open set \( O \), the set \( f^{-1}(O) \) is in \( \mathcal{B} \). You may use the fact that \( \mathcal{B} \) is a \( \sigma \)-algebra.)
4. Let \( A \subset \mathbb{R} \) be countable. Show that \( \lambda^*(A) = 0 \).
5. Let \( f \) and \( g \) be nonnegative \( \mathcal{M} \)-measurable functions and let \( E \in \mathcal{M} \). Use the definition of the Lebesgue integral to show that if \( f(x) \leq g(x) \) for all \( x \in E \) then \( \int_E f \, d\lambda \leq \int_E g \, d\lambda \).
6. Suppose that \( E \in \mathcal{M} \) with \( \lambda(E) < \infty \). Show that for every \( \varepsilon > 0 \) there is an open set \( O \) such \( E \subset O \) and \( \lambda(O \setminus E) < \varepsilon \). Show that the condition \( \lambda(E) < \infty \) can be dropped.
7. A subset \( G \subset \mathbb{R} \) is said to be a \( G_\delta \) if there is a sequence of open sets \( \{O_n\}_{n=1}^\infty \) such that \( G = \bigcap_{n=1}^\infty O_n \). Show that a \( G_\delta \) set must be Borel. Is every closed set \( G_\delta \)? Use the above exercise to show that for any \( E \in \mathcal{M} \), there is a \( G_\delta \) set \( G \) such that \( E \subset G \) and \( \lambda(G \setminus E) = 0 \).
8. Show that a subset \( E \subset \mathbb{R} \) is Lebesgue measurable iff there is a Borel set \( F \) such that \( E \subset F \) and \( \lambda^*(F \setminus E) = 0 \).
9. Let \( A \subset \mathbb{R} \) be a set which satisfies the Carathéodory criterion and let \( y \in \mathbb{R} \). Show that

\[
a + y = \{x + y : x \in A\}
\]

also satisfies the Carathéodory criterion. Show that \( \lambda(A) = \lambda(A + y) \).

10. Suppose \( \{x_n\}_{n=1}^\infty \) is a sequence of distinct real numbers. Set \( f_n = \chi_{[x_n, \infty)} \) and show that the function \( f : \mathbb{R} \to \mathbb{R} \) defined by \( f(x) = \sum_{n=1}^\infty 2^{-n} f_n(x) \) is nondecreasing. Show that \( f \) is \( \mathcal{M} \)-measurable; at which points is \( f \) continuous?
11. Show that a monotone function is Borel measurable.
12. Suppose that the function \( f : \mathbb{R} \to \mathbb{R} \) is continuous and nonnegative. Show that \( f \) is \( \mathcal{M} \)-measurable (according to Definition 3.11) and that \( \int_a^b f(x) \, dx = \int_E f \, d\lambda \) where \( E = [a, b] \).
13. Let \( f : \mathbb{R} \to \mathbb{R} \) be a nonnegative Lebesgue measurable function (i.e. \( f \in \mathcal{L}^+ \)) which is bounded. Show that then for every \( \varepsilon > 0 \), there is a nonnegative simple function \( g \) such that \( |f(x) - g(x)| < \varepsilon \) for all \( x \in \mathbb{R} \).
15. Let \( E_t = [0, t) \) for \( 0 < t \leq 1 \) and define the functions \( f \) and \( f_n \) for each \( n \in \mathbb{N} \) by

\[
f(x) = \begin{cases} 
\frac{1}{1-x} & \text{for } 0 \leq x < 1 \\
0 & \text{otherwise}
\end{cases}\quad \text{and} \quad f_n(x) = \begin{cases} 
1 + x + \ldots + x^n & \text{for } 0 \leq x < 1 \\
0 & \text{otherwise}
\end{cases}
\]

Show that

\[
\int_{E_t} f \, d\lambda = \lim_{n \to \infty} \int_{E_t} f_n \, d\lambda.
\]