Knots and Links

Representations of the fundamental group

We will now consider homomorphisms from knot groups into other groups. One problem with fundamental groups is that we easily arrive at presentations for them, but the presentations depend on the projection of the knot into a plane (the knot diagram). Two diagrams may represent equivalent knots, but the presentations for the fundamental group may look entirely different. There is no algebraic technique to determine whether two finitely presented groups are isomorphic. The set of homomorphisms from a group \( \pi \) into some group \( G \) that we understand well is easier to work with, and gives us a way to prove different fundamental groups are not isomorphic.

Let \( V \) be a vector space over \( \mathbb{R} \) (or \( \mathbb{C} \)). Let \( \text{Aut}(V) \) denote the set of invertible linear maps from \( V \) to \( V \). If we fix a basis, then we can identify \( V \) with \( \mathbb{R}^n \) (or \( \mathbb{C}^n \)), and we can identify \( \text{Aut}(V) \) with \( GL(n, \mathbb{R}) \) (or \( GL(n, \mathbb{C}) \)), the set of invertible \( n \times n \) matrices. This is always an acceptable way to think about \( V \) (you can, if you wish, always choose a basis) and \( \text{Aut}(V) \), but you should be careful about what concepts or quantities are basis dependent.

A representation of a group \( \pi \) on \( V \) is a homomorphism \( \rho : \pi \to \text{Aut}(V) \). Often we’ll consider representations that fix some additional structure on \( V \), such as an inner product. We’ll also use the term representation for any homomorphism into any other group \( G \). In all cases we consider, there exists a faithful representation of \( G \) on some vector space (i.e. a homomorphism \( G \to \text{Aut}(V) \)) where no nontrivial element of \( G \) acts trivially on \( V \), although we won’t necessarily make this explicit. Set \( \text{Hom}(\pi, G) = \{ \rho : \pi \to G \mid \rho \text{ is a homomorphism} \} \).

Recall that the abelianization of a group \( \pi \) is \( \pi/[\pi, \pi] \), which is defined to be the quotient of \( \pi \) by the least normal subgroup generated by the set of all commutators in \( \pi \). If \( X \) is a topological space, then the abelianization \( \pi_1(X)/[\pi_1(X), \pi_1(X)] \) is called the first homology group of \( X \).

Exercise A1. Show that if \( G \) is abelian, then any homomorphism \( \rho : \pi \to G \) factors through \( \pi \to \pi/[\pi, \pi] \).

Exercise A2. If \( \pi = \pi_1(S^3 - K) \), what is \( \pi/[\pi, \pi] \)? Hint: What happens to the Wirtinger presentation if we abelianize?

Exercise A3. Show that if \( f : G \to H \) is a homomorphism, then there is an induced map \( f_* : \text{Hom}(\pi, G) \to \text{Hom}(\pi, H) \) for any group \( \pi \).

Exercise A4. Show that if \( f : \pi \to \pi' \) is a homomorphism, then there is an induced map \( f^* : \text{Hom}(\pi', G) \to \text{Hom}(\pi, G) \) for any group \( G \).

Example. Let \( i : T^2 \to S^3 - K \) be the inclusion of the boundary into the knot complement. There is a map \( i_* : \pi_1(T^2) = \mathbb{Z} \oplus \mathbb{Z} \to \pi_1(S^3 - K) \). Thus there is a corresponding map we’ll call \( i^* : \text{Hom}(\pi_1(S^3 - K), G) \to \text{Hom}(\pi_1(T^2), G) \).

Here is what \( i^* \) is, in concrete terms. \( \pi_1(T^2) = \langle \lambda, \mu \mid [\lambda, \mu] = 1 \rangle = \mathbb{Z} \oplus \mathbb{Z} \).
\[ i^*(\rho) : \mathbb{Z} \oplus \mathbb{Z} \to G \] is simply the map sending the two generators to \( \rho(\lambda) \) and \( \rho(\mu) \). Technically, we should have some way to distinguish "the homotopy class of the longitude in \( S^3 - K \)" from "the homotopy class of the longitude in \( T^2 \)." For a brief moment, I will make this explicit with a subscript, and from then on I will count on all of you to figure this out from context. To be completely precise,

\[ (i^*(\rho))( [\lambda]_{\pi_1(T^2)} ) = \rho ( [\lambda]_{\pi_1(S^3 - K)} ) . \]

**Exercise A5.** Recall how a finitely presented group

\[ \pi = \langle x_1, \ldots, x_n \mid r_1(x_1, \ldots, x_n), \ldots, r_m(x_1, \ldots, x_n) \rangle \]

is defined in terms of free groups and quotients. Given such a group \( \pi \) and another group \( G \), how can you describe a homomorphism \( \rho : \pi \to G \)? More concretely, identify \( \text{Hom}(\pi, G) \) with a subset of \( G^k \) for some \( k \).

**Exercise A6.** If \( \rho : \pi \to G \) and \( g \in G \), then \( g \rho g^{-1} : \pi \to G \) defines another homomorphism. Here, \( g \rho g^{-1} \) is defined to be the map \( a \in \pi \mapsto g \rho(a) g^{-1} \in G \).

From Exercises A1 and A2, we are not going to be able to tell the difference between any knots by considering homomorphisms into abelian groups, such as \( \mathbb{Z}, \mathbb{Z}_n, \mathbb{R}^k \), or \( S^1 \). So let us review some of the other groups you have (or haven’t) heard of.

**The symmetric group \( S_n \).**

This is the set of permutations on \( n \) elements, containing \( n! \) elements. Note that the cyclic permutation

\[
\begin{pmatrix}
1 & 2 & \cdots & n - 1 & n \\
2 & 3 & \cdots & n & 1
\end{pmatrix}
\]

has order \( n \). If you like, you can think of \( S_n \) acting on \( \mathbb{R}^n \) by permuting the coordinates of vectors, e.g.,

\[
\begin{pmatrix}
1 & 2 & 3 \\
2 & 3 & 1
\end{pmatrix}
(x_1, x_2, x_3) = (x_3, x_1, x_2).
\]

This permutation could also be represented by the matrix

\[
\begin{bmatrix}
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix}.
\]

In fact, here is a more interesting fact about the \( S_n \) action on \( \mathbb{R}^n \). Scalar multiples of \((1, 1, \ldots, 1)\) are all fixed by the entire group of permutations. The \( n - 1 \) dimensional plane given by \( x_1 + \cdots + x_n = 0 \) is invariant under the \( S_n \) action (vectors in this plane are taken to vectors in this plane), and there are no invariant proper subspaces.

**Exercise B1.** What is the center of \( S_n \)? What are the elements of \( S_5 \) of order 5? What are the elements of order 4? What are the elements of order 2 (ironically, this is a little trickier)?
Exercise B2. If \( p \) and \( q \) are relatively prime integers, then the fundamental group \( \pi = \pi_1(S^3 - T(p, q)) \) of the complement of the \((p, q)\) torus knot has a presentation \( \pi = \langle a, b \mid a^p = b^q \rangle \).

a. Show that if \( \rho : \pi \to G \) is a representation, then \( \rho(a^p) \) is central in \( G \).
b. What is the center of \( S_3 \)? What are the "\( p \)th roots" of elements of the center?
c. Describe the elements of \( \text{Hom}(\pi_1(S^3 - T(4, 5)), S_3) \) as completely as you can. How many elements are there? How many elements are there in \( \text{Hom}(\pi_1(S^3 - T(2, 5)), S_3) \)? Note that if these numbers are different, this proves that \( T(2, 5) \) and \( T(4, 5) \) are not equivalent.
d. How many elements of \( \text{Hom}(\pi_1(S^3 - T(11, 13)), S_4) \) are there? Can you prove a fairly strong theorem about which torus knots are not equivalent to one another without too much combinatorics?

The dihedral group \( D_n \).
Define the set \( D_n \) to be \( \mathbb{Z}_n \times \mathbb{Z}_2 \). But define multiplication in the following way.

\[
([k], [i])([\ell], [j]) = ([k + (-1)^i j], [i + j]).
\]

Single square brackets mean "\( \text{mod } n \)" and double brackets mean "\( \text{mod } 2 \)." You should interpret this group as the symmetries of the regular \( n \)-gon in the (complex) plane with vertices \( 1, e^{\frac{2\pi i}{n}} \), \ldots, \( e^{\frac{2\pi i(n-1)}{n}} \) as follows.

\[
([k], [i])z = \begin{cases} 
  e^{\frac{2\pi i k}{n}} z & i = 0 \pmod{2} \\
  e^{\frac{2\pi i k}{n}} z & i = 1 \pmod{2}.
\end{cases}
\]

Note that there are two groups of symmetries, rotations by \( \frac{2\pi k}{n} \), and complex conjugation. And as a set this is exactly the set of symmetries generated by all compositions. But since the two group actions on \( \mathbb{C} \) don’t commute, it is not a product group.

Exercise C1. Draw a knot diagram, label it the Wirtinger generators \( x_1, \ldots, x_m \) and write down the Wirtinger relations. Let \( X_1, \ldots, X_m \) be elements of \( D_n \) (think of these as potential images \( \rho(x_i) = X_i \) for a representation).

a. What condition on \( X_1, \ldots, X_m \) does each Wirtinger relation impose?

Hints:

i. What is the inverse of \( ([k], [i]) \)?
ii. What is \( ([k], [i])([\ell], [j])([k], [i])^{-1} \)?
b. There is a knot invariant called "\( n \)-colorability." It has to do with whether you can label the arcs in the knot diagram with element of \( \mathbb{Z}_n \) satisfying a certain condition. (With this definition of \( n \)-colorability, it is necessary to prove that the property is independent of knot diagram). By this point, what is the condition, how is it related to \( D_n \) representations of the fundamental group, and how do you know that it is a topological invariant?
c. Check some knots for \( 3 \)-colorability and \( 5 \)-colorability.
The groups $O(n)$ and $SO(n)$. 
Consider $\mathbb{R}^n$ with the standard inner product, $(x_1, \ldots, x_n) \cdot (y_1, \ldots, y_n) = \sum x_i y_i$. Note that we can define length and angles in terms of the dot product, $\|x\| = \sqrt{x \cdot x}$ and $\theta = \cos^{-1} \left( \frac{x \cdot y}{\|x\| \|y\|} \right)$.

Define

$$O(n) = \{ g \in GL(n, \mathbb{R}) | g g^T = I d \}.$$ 

Exercise D1. Show by writing these out as sums over indices, as in

$$[a_{ij}] \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} \sum_i a_{i1} v_i \\ \sum_i a_{i2} v_i \\ \vdots \\ \sum_i a_{in} v_i \end{bmatrix}$$

that $\langle A \vec{v}, \vec{w} \rangle = \langle \vec{v}, A^T \vec{w} \rangle$ for any $m \times n$ matrix $A$. Use this fact about transposes to show that $g \in O(n)$ iff $g$ preserves all inner products, i.e. $\langle \vec{x}, \vec{y} \rangle = \langle g \vec{x}, g \vec{y} \rangle$.

Note that $O(n) \subset \mathbb{R}^n$, so it makes sense to take derivatives of a path of matrices. Let $g(t) = [a_{ij}(t)]$. Define $g'(t) = [a'_{ij}(t)]$.

Exercise D2. Show that if $g(t)$ and $h(t)$ are paths in $GL(n, \mathbb{R})$ then $(g(t) h(t))' = g'(t) h(t) + g(t) h'(t)$.

Exercise D3. Show that if $g(t)$ is a path in $O(n)$ with $g(1) = I d$ then $g'(0)$ is skew-symmetric. (Differentiate the defining relation for $O(n)$.)

Exercise D4. Show that if $g \in O(n)$ then $\det g = \pm 1$. Hint: What is $\det(AB)$? Apply this to the defining relation.

We define $SO(n)$ to be

$$SO(n) = \{ g \in O(n) | \det(g) = 1 \}.$$ 

Here is a nice, not so obvious fact:

Every element of $SO(3)$ is a rotation about some axis (by some angle) in $\mathbb{R}^3$. That is, up to change of basis, it looks like

$$\begin{bmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$ 

The proof involves diagonalizing over $\mathbb{C}$. The eigenvalues are unit complex numbers that solve a real polynomial equation of degree 3, so one is real, hence is $\pm 1$. One can rule out $-1$ because of the determinant condition.

Exercise D5. 
Show that in addition to the condition in D3, tangent vectors to $SO(n)$ at the identity are traceless. Hint: write $g(t) = I d + h(t)$ where $h(t)$ is a path of matrices with $h(0) = 0$. You need not write out the whole determinant of $I d + h(t)$ to figure out what the “first order” terms are (those that contribute to $\frac{d}{dt} \det(g(t)))|_{t=0}$).
The exponential map, the eighth wonder of the modern world.

Let \( gl(n, \mathbb{R}) \) denote the set of all \( n \times n \) real matrices. Suppose that there is a map

\[
\exp : gl(n, \mathbb{R}) \to gl(n, \mathbb{R}).
\]

What properties should this map have for us to legitimately call it the exponential map? For what \( u \in gl(n, \mathbb{R}) \) should \( \exp(u) \) be invertible? What should the inverse be, if it exists. What formula do you expect for a product of two exponentials? What should the derivative of \( \exp(tu) \) with respect to \( t \) be at \( t = 0 \)? The key to proving all these things is the following.

Exercise e (what else?).

Write down the proof that the Taylor series for \( e^x \) converges for all \( x \). Replace \( x \) by the matrix \( u \). Show the series of matrices converges. You’ll need the following.

A norm on a vector space \( V \) is a map \( \| \cdot \| : V \to \mathbb{R} \) satisfying \( \| v \| \geq 0 \) and \( \| v \| = 0 \) iff \( v = 0 \), \( \| cv \| = |c| \| v \| \) for \( c \in \mathbb{R} \), and the triangle inequality \( \| v + w \| \leq \| v \| + \| w \| \). Given an inner product on \( V \), we can define \( \| v \| = \sqrt{v \cdot v} \).

Viewing matrices in \( gl(n, \mathbb{R}) \) as linear operators \( L : \mathbb{R}^n \to \mathbb{R}^n \), we can put the operator norm on \( gl(n, \mathbb{R}) \) by

\[
\| A \| = \sup_{v \neq 0} \frac{\| Av \|}{\| v \|}.
\]

Check that the operator norm satisfies the condition that \( \| AB \| \leq \| A \| \| B \| \).

Now you can finish of Exercise e.

Now, verify all but one of the following properties for \( \exp \) by writing everything in Taylor series. One of these doesn’t work.

- \( \exp 0 = 1 \).
- \( \frac{d}{dt} \exp(tu) = u \exp(tu) \).
- \( \exp(u) \exp(u) = \exp(2u) \).
- \( \exp(u) \exp(v) = \exp(u + v) \).
- \( \frac{d}{dt} \det(\exp(tu))|_{t=0} = \text{Tr}(u) \).
- If \( u \) is skew-symmetric then \( \exp(u) \) is in \( O(n) \).

At least as interesting, and in some ways simpler, are the complex analogs of \( O(n) \) and \( SO(n) \), which we’ll go into next. One reason why they are simpler is that all elements can be diagonalized. Before we take this detour into complex linear algebra, let me discuss one application of fundamental groups of knots.
Fundamental groups and 3-manifolds

Definition A closed 3-manifold is a compact topological space $X$ with the property that each $x \in X$ has a neighborhood $x \in X$ which is homeomorphic to an open ball in $\mathbb{R}^3$. (The term closed means compact and without boundary. In a 3-manifold with boundary, some points are allowed to have neighborhoods homeomorphic to the open ball intersected with the closed upper half space. These points make up the boundary of the 3-manifold.)

Some familiar examples are $S^3$ and $\mathbb{R}P^3 = S^3/\{x \sim -x\}$. Let me give you a less familiar example.

Lens space $L(p, q)$.
Let $p$ and $q$ be two integers, relatively prime to one another. View $S^3$ as a subset of $\mathbb{C}^2$, i.e.

$$S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\}.$$

Define an equivalence relation on $S^3$ by

$$(z_1, z_2) \sim \left(e^{\frac{2\pi i}{p} z_1}, e^{\frac{2\pi i}{q} z_2}\right).$$

$L(p, q)$ is the quotient $S^3/\sim$.

Theorem
Each of the quotient maps $S^3 \to \mathbb{R}P^3$ and $S^3 \to L(p, q)$ is a covering map.

Questions

i. Why is the theorem true?

ii. How many points are in the preimage of a point $x_0 \in S^3$?

iii. What can we conclude about $\pi_1(\mathbb{R}P^3)$ and $\pi_1(L(p, q))$?

A history lesson
In 1900, Henri Poincaré proved the following theorem.

Theorem
Any closed 3-manifold $X$ with $H_1(X) = \pi_1(X)/[\pi_1(X), \pi_1(X)]$ trivial is homeomorphic to $S^3$.

Unfortunately, within a couple years, Poincaré and others had found counterexamples. Here is Poincaré’s counterexample, which is now known as the Poincaré homology sphere. Dehn surgery on a knot $K \subset S^3$ is the process of removing a tubular neighborhood of the knot and pasting in a solid torus by a perhaps different homeomorphism of the boundary torus. I’ll describe this identification map by explaining where the longitude and meridian on the solid torus end up on the boundary of $S^3 - K$. (See Rolfsen, 9D.)
For the purposes of applying the Seifert Van Kampen Theorem, it is easier to
give generators of \( \pi_1(T^2) \) and explain what the two maps \( i_1 : T^2 \to S^3 - K \)
and \( i_2 : T^2 \to S^1 \times D^2 \) do to these generators.

\[
\pi_1((S^3 - K) \cup_h S^1 \times D^2) = \langle x, y, z, b \mid xy = yz = zx, i_1*(m) = i_2*(m), i_1*(l) = i_2*(l) \rangle \\
= \langle x, y, z, b \mid z = xy^{-1}, xy = yxy^{-1}, z^{-2}xy = 1, b = xyz \rangle \\
= \langle x, y \mid yxy = x, (xyx^{-1})^2(x)(xyx^{-1})y = 1 \rangle \\
= \langle xyx = yxy, xy^{-2}xy^{-1}y = 1 \rangle.
\]

Let’s see what the abelianization is. Let \( X = [x] \) and \( Y = [y] \) be the equivalent classes modulo the commutator subgroup. We’ll use additive notation for the operation in the abelianized group. That is, \([xy] = [x] + [y] \).

Then the equations for \( X, Y \) are \( 2X + Y = X + 2Y \) and \( X - 2Y + X + Y - X + Y = 0 \). These easily reduce to \( X = Y \) and \( X = 0 \). This shows that this manifold is has trivial first homology group.

To see that this manifold is not actually \( S^3 \), one option would be to find a nontrivial homomorphism of \( \pi_1 \) into some group. We’ll be able to see this, but I believe it requires a different group than the ones discussed above. So we’ll postpone this part for later.
**Poincaré Conjecture** 1904
Any closed 3-manifold with trivial fundamental group is homeomorphic to the 3-sphere.

The current status of this conjecture is a little up in the air. Over a year ago, Perlmann claimed he had a proof of this. It has been looked at by mathematicians since then, and no one has found a flaw in it. So far, all gaps people have been found have been filled by Perlmann. But it seems his argument is sufficiently involved that there are not yet experts willing to proclaim they’ve understood it and that it is correct.

The following notion was motivated by the 100 year long search for counterexamples to the Poincaré conjecture.

**Property P**
We say a knot has *Property P* if every nontrivial surgery along $K$ yields a non-simply-connected 3-manifold (i.e. no surgery on this knot can provide a counterexample to the conjecture). Rolfsen states (p. 280) that it is a conjecture that all nontrivial knots have *Property P*. This conjecture was proven by Peter Kronheimer and Tomasz Mrowka (my thesis advisor) November 2003. What Kronheimer and Mrowka actually prove is that if $X$ is a nontrivial surgery on a nontrivial knot, then $\pi_1(X)$ admits a representation into the set of $2 \times 2$ complex invertible matrices with nonabelian image. Hence $\pi_1(X)$ is nonabelian. The proof of Kronheimer and Mrowka was very indirect and used a lot of other technology. A direct proof would be very satisfying.