You may bring one sheet of notes with definitions and statements of theorems. It should not contain any proofs or worked out problems, and it should be hand-written.

The test will cover up through Theorem 3.17 (Monotone Convergence Theorem) from § 3.6. Look over the following homework problems. The list includes three probability measure problems from § 4.1. Solutions to some of these are available at http://wolfweb.unr.edu/homepage/naik/classes/713/

Section 1.2: 1.15, 1.21, 1.23
Section 2.6: 2.79, 2.83, 2.89
Section 3.2: 3.15, 3.16, 3.18
Section 3.3: 3.24, 3.25, 3.26
Section 3.4: 3.30-3.38, 3.44, 3.45, 3.47a, 3.50a
Section 3.5: 3.53, 3.56a, 3.62
Section 3.6: 3.71
Section 4.1: 4.3, 4.7, 4.9

See below for additional practice questions.

1. Show that a nonempty set \( A \) is countable iff there is a surjective function \( f : \mathbb{N} \to A \).

2. Let \( \Omega \) be a nonempty countable set and suppose that \( A \) is a \( \sigma \)-algebra of subsets of \( \Omega \) such that for any pair of distinct elements \( x, y \in \Omega \) there is an \( A \in \mathcal{A} \) such that \( x \in A \) but \( y \notin A \). Show that \( \mathcal{A} = \mathcal{P}(\Omega) \).

3. Give an example (if possible) of a \( \sigma \)-algebra which is not an algebra.

4. Let \( f : \mathbb{R} \to \mathbb{R} \) be defined by \( f(x) = x^3 \). Use the definition of continuity to prove that \( f \) is continuous.

5. Let \( f : X \to Y \) be a function and suppose that \( A \) and \( B \) are subsets of \( Y \). Show that \( f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B) \), \( f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B) \), and \( f^{-1}(A^c) = f^{-1}(A)^c \).

6. Suppose that \( \{x_n\}_{n=1}^{\infty} \) is a sequence of real numbers which converges to \( x_0 \) and that \( f : \mathbb{R} \to \mathbb{R} \) is continuous at \( x_0 \). Show that \( f(x_n) \to f(x_0) \) as \( n \to \infty \).

7. Let \( \{x_n\}_{n=1}^{\infty} \) be a sequence of real numbers show that \( \limsup x_n = \infty \) iff \( \sup_n x_n = \infty \) and that \( \liminf x_n = \infty \) iff \( \lim_n x_n = \infty \).

8. Let \( f \) and \( g \) be real valued continuous functions defined on \( \mathbb{R} \) show that \( f \vee g \) is continuous.

9. Suppose that \( f : X \to Y \) is a function and \( \mathcal{A} \) is a \( \sigma \)-algebra of sets on \( Y \). Show that the following collection of subsets of \( X \) is also a \( \sigma \)-algebra

\[ \{f^{-1}(A) : A \in \mathcal{A}\} \]

10. Define what it means for a function to be simple. Show that the collection of all simple functions forms an algebra.

11. Let \( A \subset \mathbb{R} \) be given. Carefully define what is meant by the outer measure \( \lambda^*(A) \). Suppose that \( B \) is another subset of \( \mathbb{R} \); use the definition to show that \( \lambda^*(A \cup B) \leq \lambda^*(A) + \lambda^*(B) \).

12. Prove that \( \chi_\mathbb{Q} \) is not Riemann integrable on \([0,1]\).

13. Show that \( f : \mathbb{R} \to \mathbb{R} \) is Borel measurable iff \( f^{-1}((r,\infty)) \in \mathcal{B} \) for all \( r \in \mathbb{Q} \).

14. Suppose that \( f : A \to B \) is one-to-one. Show that \( A \) is countable if \( B \) is. Does the converse hold?

15. Suppose that \( f : A \to B \) is a surjective function. Show that there is a function \( g : B \to A \) such that \( f(g(b)) = b \) for all \( b \in B \). Must \( g \) be one-to-one?

16. Let \( A \subset \mathbb{R} \) be countable. Show that \( \lambda^*(A) = 0 \).

17. Let \( \{x_n\}_{n=1}^{\infty} \) and \( \{y_n\}_{n=1}^{\infty} \) be two sequences of real numbers. Show that \( \limsup(x_n + y_n) \leq \limsup x_n + \limsup y_n \) if the right hand side makes sense.

18. Let \( f \) and \( g \) be nonnegative \( \mathcal{M} \)-measurable functions and let \( E \in \mathcal{M} \). Use the definition of the Lebesgue integral to show that if \( f(x) \leq g(x) \) for all \( x \in E \) then \( \int_E f \, d\lambda \leq \int_E g \, d\lambda \).

19. Suppose that \( E \in \mathcal{M} \) with \( \lambda(E) < \infty \). Show that for every \( \varepsilon > 0 \) there is an open set \( O \) such that \( E \subset O \) and \( \lambda(O \setminus E) < \varepsilon \). Show that the condition \( \lambda(E) < \infty \) can be dropped.

20. A subset \( G \subset \mathbb{R} \) is said to be \( G_\delta \) if there is a sequence of open sets \( \{O_n\}_{n=1}^{\infty} \) such that \( G = \cap_{n=1}^{\infty} O_n \). Show that a \( G_\delta \) set must be Borel. Is every closed set \( G_\delta \)? Use the above exercise to show that for any \( E \in \mathcal{M} \), there is a \( G_\delta \) set \( G \) such that \( E \subset G \) and \( \lambda(G \setminus E) = 0 \).

21. Show that a subset \( E \subset \mathbb{R} \) is Lebesgue measurable iff there is a Borel set \( F \) such that \( E \subset F \) and \( \lambda^*(F \setminus E) = 0 \).

22. Let \( \{x_n\}_{n=1}^{\infty} \) and \( \{y_n\}_{n=1}^{\infty} \) be two sequences of real numbers and suppose that \( x_n \leq y_n \) for all \( n \in \mathbb{N} \). Show that \( \liminf x_n \leq \liminf y_n \).
(23) Let $A \subset \mathbb{R}$ be a set which satisfies the Carathéodory criterion and let $y \in \mathbb{R}$. Show that

$$A + y = \{ x + y : x \in A \}$$

also satisfies the Carathéodory criterion. Show that $\lambda(A) = \lambda(A + y)$.

(24) Show that every bounded sequence of real numbers has a convergent subsequence.

(25) Prove that a finite union of closed sets is closed.

(26) Suppose $\{x_n\}_{n=1}^{\infty}$ is a sequence of distinct real numbers. Set $f_n = \chi_{[x_n, \infty)}$ and show that the function $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = \sum_{n=1}^{\infty} 2^{-n} f_n(x)$ is nondecreasing. Show that $f$ is $\mathcal{M}$-measurable; at which points is $f$ continuous?

(27) Show that a monotone function is Borel measurable.

(28) Suppose that the function $f : \mathbb{R} \rightarrow \mathbb{R}$ is continuous and nonnegative. Show that $f$ is $\mathcal{M}$-measurable and that

$$\int_a^b f(x) \, dx = \int_E f \, d\lambda,$$

where $E = [a, b]$.

(29) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a nonnegative Lebesgue measurable function (i.e. $f \in \mathcal{L}^+$) which is bounded. Show that then for every $\varepsilon > 0$, there is a nonnegative simple function $g$ such that $|f(x) - g(x)| < \varepsilon$ for all $x \in \mathbb{R}$.

(30) Carefully state the Monotone Convergence Theorem and prove two interesting corollaries.

(31) Let $E_t = [0, t)$ for $0 < t \leq 1$ and define the functions $f$ and $f_n$ for each $n \in \mathbb{N}$ by

$$f(x) = \begin{cases} \frac{1}{1-x} & \text{for } 0 \leq x < 1 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad f_n(x) = \begin{cases} 1 + x + \cdots + x^n & \text{for } 0 \leq x < 1 \\ 0 & \text{otherwise}. \end{cases}$$

Show that

$$\int_{E_t} f \, d\lambda = \lim_{n \to \infty} \int_{E_t} f_n \, d\lambda.$$