Equivariant Concordance of Knots in $S^3$

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1 Introduction

Two knots are said to be equivariant concordant if they cobound a periodic concordance. A natural question is how to determine if two concordant, periodic knots are equivariant concordant. Equivariant concordance of higher dimensional knots (isovariant cobordism) was studied in 1991 by R. Cruz [5, Section 3]. The results therein do not apply to knots in $S^3$. In this paper we discuss equivariant concordance of classical knots, i.e., of knots in $S^3$.

We observe that an obstruction to two period $q$, concordant knots being equivariant concordant is given by the linking number between a knot and its axis of periodicity (Corollary 3.2). This linking number is determined by the Alexander polynomial of the knot (see [14]), and is easily computable. Using this we give several examples of concordant, periodic knots with 10 or fewer crossings which are not equivariant concordant. (See Examples 3.4, 3.5.) Combining the linking number condition of Corollary 3.2 with [6, Corollary 5] for a genus $g$ knot which is equivariant slice with period $q$, we

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have \( q \leq g + 1 \). On the other hand, infinitely many examples of equivariant slice knots, i.e., of knots which are equivariant concordant to the trivial knot, can be obtained by constructing equivariant ribbon disks immersed in \( S^3 \). We show one such family of knots in Example 2.5. In the rest of the paper we discuss equivariant slice knots.

A periodic knot bounds an equivariant Seifert surface. We show that the Seifert form corresponding to such a surface for an equivariant slice knot has an equivariant metabolizer (Theorem 4.3). Combining this with the works of A. Casson, C. Gordon [2] and P. Gilmer [9], we have Theorem 5.2 which states that if \( M \) is a prime power fold cyclic cover of \( S^3 \) branched over an equivariant slice knot, then \( H_1(M) \) has a subgroup \( G \) with \( |G| = \sqrt{|H_1(M)|} \) which is invariant under the lift of the periodic map. Furthermore, for a prime \( p \) and a non-negative integer \( r \), the Casson–Gordon invariants \( \tau(K, \chi) \) vanish for all \( \chi: H_1(M) \to C_p^r \) with \( \chi|_G \equiv 1 \). Here \( C_p^r \) denotes the cyclic group of order \( p^r \).

Finally it follows from a transfer argument that the fixed point set of the map induced by the lift of the periodic map on \( H_1 \) of a cyclic cover of \( S^3 \) branched over a periodic knot is isomorphic to \( H_1 \) of the corresponding cover of \( S^3 \) branched over the quotient knot (Lemma 5.1). We use this to obtain conditions on Alexander polynomials of equivariant slice knots in Theorem 5.3.

The paper is organized as follows. In Section 2 we give definitions. We also give examples of equivariant slice (ribbon) knots. In Section 3 we discuss linking numbers and show that certain concordant, periodic knots are not equivariant concordant. In Sections 4 and 5 we restrict our attention to equivariant slice knots. In Section 4 we discuss metabolizers and prove that equivariant slice knots have equivariant metabolizers. In Section 5 we obtain criteria for equivariant sliceness using cyclic branched covers. Theorem 5.2 gives the equivariant version of [2, Theorem 2]. In Theorem 5.3 we
obtain conditions on Alexander polynomials of equivariant slice knots using homology of cyclic branched covers.

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\section{Definitions and Examples}

We begin by defining a periodic knot.

\textbf{Definition 2.1} A knot \(K\) in \(S^3\) is said to be periodic if there exists an integer \(q > 1\) and an orientation preserving diffeomorphism \(f : S^3 \to S^3\) such that \(f(K) = K\), \(\text{order}(f) = q\), and the fixed point set of \(f\), which we will call the axis of periodicity, is a circle disjoint from \(K\). Any such \(q\) is called a period of \(K\), and any such \(f\) the corresponding periodic transformation.

It follows from the positive solution to the Smith conjecture (see \([13]\)) that the axis of \(f\) is unknotted. Equivalently, the action of \(f\) on \(S^3\) is conjugate by diffeomorphisms of \(S^3\) to an orthogonal action (see \([12]\)). It follows that the quotient manifold \(S^3/\langle f \rangle\) is diffeomorphic to \(S^3\). The image of the knot \(K\) in the quotient manifold is called the quotient knot, denoted by \(\overline{K}\).

\textbf{Definition 2.2} Two knots are concordant if they cobound a smooth 2-manifold \(C \cong S^1 \times I\) properly embedded in \(S^3 \times I\). A knot which is concordant to the unknot is called a slice knot.

It is easy to see that a knot \(K\) is slice if and only if it bounds a properly embedded smooth 2-disk \(D\) in the 4-ball \(B^4\).
Definition 2.3 Two period $q$, concordant knots $K_1$ and $K_2$ are equivariant concordant or $q$-equivariant concordant if there are period $q$ transformations $f_i$ for $K_i$, $1 \leq i \leq 2$, and an order $q$ diffeomorphism $f$ of $S^3 \times I$ which leaves the concordance $C$ invariant and restricts to $f_i$ on the boundary component containing $K_i$. A period $q$ knot is called equivariant slice if it is equivariant concordant to the unknot.

Equivalently, a knot is equivariant slice if it is slice as well as periodic with period $q$, and the periodic map of $S^3$ extends to a period $q$ self-diffeomorphism of $B^4$ which preserves the slicing disk $D$.

If a knot $K = h(S^1)$ where $h$ is an immersion of the 2-disk $D^2$ in $S^3$ which has the property that each component of self intersection is an arc $A \subset h(D^2)$ for which $h^{-1}(A)$ consists of two arcs in $D^2$, one of which is in the interior, then $K$ is called a ribbon knot. Ribbon knots are slice. We will call such a $K$ in $S^3$ equivariant ribbon if it is periodic with period $q$ and $h(D^2)$ is invariant under the periodic transformation. Note that since $K$ is disjoint from the fixed point set of the action, none of the arcs of self-intersection of $h(D^2)$ are pointwise fixed by the periodic map. The following result is easy to see.

Proposition 2.4 Equivariant ribbon knots are equivariant slice. 

In Example 2.5 we give a family of equivariant ribbon knots. This family of knots was constructed in [17] and pointed out to me by Ted Stanford. For the knot in Figure 1 all Vassiliev invariants of order $\leq n$ vanish.
Example 2.5  The knot $K_n$ shown in Figure 1 is $n$-equivariant ribbon.

\[ K_n \]

**Figure 1**

The knot $6_1$ (in the notation of [18]) equals $K_2$ and is 2-equivariant ribbon; $9_{41}$ is $K_3$ and so it is 3-equivariant ribbon.

\[ K_2 = 6_1 \quad K_3 = 9_{41} \]

**Figure 2**
3 Linking Numbers

It is natural to ask whether all concordant periodic knots are equivariant concordant. In this section we discuss some obstructions to equivariant concordance.

**Proposition 3.1** If a period $q$ knot $K_1$ with axis $P_1$ is equivariant concordant to a period $q$ knot $K_2$ with axis $P_2$, then the links $K_1 \cup P_1$ and $K_2 \cup P_2$ are concordant.

**Proof:** Let $f$ be the order $q$ diffeomorphism of $S^3 \times I$ as in Definition 2.3. Let $C$ be the concordance between $K_1$ and $K_2$ which is invariant under $f$. Let $B$ denote $\text{Fix}(f)$. It follows that $B \cong S^1 \times I$ (see [19]). By the tubular neighborhood theorem (see [1]) $B$ has an invariant neighborhood diffeomorphic to $B \times B^2$ on which the action of $f$ is standard. It follows that $B \cap C$ consists of isolated points. Let $n = \#(B \cap C) = \#\text{Fix}(f|_C)$. Then $C$ is a cover of $\overline{C} = C/\langle f \rangle$ branched over $n$ points. The Riemann-Hurwitz formula tells us that $\chi(C) = q\chi(\overline{C}) - n(q-1)$. This implies $g(C) = qg(\overline{C}) + \frac{n(q-1)}{2}$. Since $g(C) = 0$, we have $n = g(\overline{C}) = 0$. Thus $B \cup C$ gives a concordance between the links $K_1 \cup P_1$ and $K_2 \cup P_2$. \hfill \Box

**Corollary 3.2** If a period $q$ knot $K_1$ with axis $P_1$ is equivariant concordant to a period $q$ knot $K_2$ with axis $P_2$, then $\text{lk}(K_1, P_1) = \text{lk}(K_2, P_2)$. \hfill \Box

**Corollary 3.3** Let $K$ be equivariant slice, and let $P$ be the axis of periodicity. Then $\text{lk}(K, P) = 1$ and $K \cup P$ is concordant to the Hopf link. \hfill \Box
We use these corollaries to show that certain knots which are concordant and periodic with the same period cannot be equivariant concordant.

The linking number \( \lambda = \text{lk}(K, P) \) appears in the Murasugi conditions on the Alexander polynomial of a periodic knot \([14]\). In particular, if \( K \) has a prime period \( q \), and \( \Delta, \overline{\Delta} \) are Alexander polynomials of \( K \) and \( \overline{K} \), respectively, then

\[
\Delta(t) \equiv (\overline{\Delta}(t))^q(1 + t + \cdots + t^{q-1})^{q-1} \mod q.
\]

For the knots in the example below we referred to the list of knot concordances given in \([4]\). We determine \( \lambda \) using the congruence condition given above.

**Example 3.4** The knots 8_9, 9_{27}, 10_{12}, and 10_75 from the tables of \([18]\) are slice, and periodic with period 2, but not equivariant slice, as the linking number \( \lambda \) equals 7 for each of these. The knot 10_{123} is slice and it has period 5, but \( \lambda = 3 \), so it is not equivariant slice.

**Example 3.5** The knots 3_1 and 9_{20} are concordant to each other and they both have period 2, but they are not equivariant concordant. For 3_1 we have \( \lambda = 3 \) but for 9_{20}, \( \lambda = 7 \).

A. Edmonds showed in \([6]\) that for a knot with genus \( g \) the possible periods are bounded above by \( 2g + 1 \), and the only possible periods which are greater than \( g \) are \( g + 1 \) and \( 2g + 1 \). Also, if the period is \( 2g + 1 \), then the linking number \( \lambda = 2 \). This gives us the following corollary for equivariant slice knots. We include the proof for the sake of completeness.
Corollary 3.6 If $K$ is $q$-equivariant slice, and it has genus $g$, then $q \leq g+1$.

Proof: It is shown in [6] that a genus $g$ periodic knot bounds an equivariant Seifert surface of genus $g$. Let $F$ be such a surface for $K$. Let $P$ be the axis of periodicity. Then the quotient of $F$ under the periodic action is a Seifert surface $\overline{F}$ for the quotient knot, and $F$ is a branched cover of $\overline{F}$ over $l$ points, where $l = \#(F \cap P)$. As $\lambda$ is the algebraic intersection number of $F$ and $P$, we have $l \geq \lambda$, and $l \equiv \lambda \mod 2$. Now by the Riemann-Hurwitz formula we have
\[ g = q \cdot g(\overline{F}) + \frac{(q - 1)(l - 1)}{2}. \]
If $g < q$, then the only solutions to this equation are (i) $q = g + 1$, $g(\overline{F}) = 0$, $l = 3$, and (ii) $q = 2g + 1$, $g(\overline{F}) = 0$, $l = 2$. Corollary 3.3 rules out the second possibility. \hfill \Box

We now restrict our attention to equivariant slice knots.

4 Equivariant Metabolizers

In 1969, J. Levine [11] showed that the Seifert form of a slice knot is metabolic, i.e., if $F$ is a Seifert surface for the knot, then there exists a direct summand of $H_1(F)$, which is self annihilating with respect to the Seifert form and has half the rank of $H_1(F)$. Such a summand is called a metabolizer of the Seifert form. If $K$ is a slice knot bounding a Seifert surface $F$, then the slice disk for $K$ and the surface $F$ cobound a 3-manifold in $B^4$. The metabolizer is the kernel of the map induced on homology by the inclusion of $F$ in this 3-manifold.

A periodic knot bounds an equivariant Seifert surface in $S^3$. We show that if $K$ is equivariant slice then the 3-manifold cobounded by the equivariant slice disk and the equivariant Seifert surface can be chosen to be equivariant.
Proposition 4.1 Let $K$ be an equivariant slice knot with a periodic map $f$ of $B^4$, an equivariant Seifert surface $F$, and an equivariant slice disk $D$. Then there is a 3-manifold $R$ in $B^4$ bounded by $D \cup F$ which is equivariant, i.e. $f(R) = R$.

Proof: The quotient manifold $M = B^4/\langle f \rangle$ is a simply connected, acyclic 4 dimensional manifold (see [1]) with boundary $S^3$. By Freedman’s work [7] it follows that it is topologically a 4-ball. (Note that there are non-standard finite group actions on $B^4$ (see [8]) and so the quotient manifold may not be diffeomorphic to the standard 4-ball.)

The images of $K$, $F$ and $D$ are the quotient knot $\overline{K}$, a Seifert surface $\overline{F}$, and a 2-disk $\overline{D}$ for $\overline{K}$, respectively. An obstruction theory argument shows that $\overline{F} \cup \overline{D}$ bounds a 3-manifold $\overline{R}$. (See [18].) This manifold can be made transverse to the quotient $\overline{B}$ of the fixed point set $B$ of the periodic transformation. Then the intersection of $\overline{R}$ and $\overline{B}$ is a set of curves. Now, $B^4$ is the $q$-fold cyclic cover of $M$ branched over $B$. The lift of $\overline{F}$ is an equivariant Seifert surface $F$ for $K$. Let $R$ be the lift of $\overline{R}$ in $B^4$. A branched cover of a compact 3-manifold with a branch set consisting of proper arcs and curves is a 3-manifold. Clearly, $\partial R = \overline{F} \cup \overline{D}$ and $f(R) = R$. \hfill \Box

We are now ready to prove that equivariant slice knots have equivariant metabolizers.

Theorem 4.2 Let $K$, $F$, $D$, and $f$ be as in Proposition 4.1. Let $\theta$ be the Seifert form. Then there exists a summand $H$ of $H_1(F)$ such that

(i) $\text{rk}(H) = \frac{1}{2}\text{rk}(H_1(F))$,
(ii) $\theta_{H \times H} = 0$, and
(iii) $(\tilde{J}_F)_\theta(H) = H$.

Such an $H$ will be called an equivariant metabolizer for $K$. 

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Proof: By Proposition 4.1 we know that there is an equivariant 3-manifold bounded by $F \cup D$. Half of the generators of $H_1(F)$ have representing cycles which bound 2-chains in $R$. It is easy to see that the summand of $H_1(F)$ spanned by these generators is a metabolizer of the Seifert form, and it is equivariant. \hfill \Box

In 1975, A. Casson and C. Gordon introduced Witt class invariants $\tau(K, \chi)$ to prove the existence of knots with metabolic Seifert forms which are not slice. (See [2]). If $M$ is the $k$-fold cyclic cover of $S^3$ branched over the knot $K$ for a prime-power $k$, then $\chi$ denotes a prime-power order $\mathbb{Q}/\mathbb{Z}$ character on $H_1(M)$. It was shown in [2] that if $K$ is a slice knot, then there exists a subgroup $G$ of $H_1(M)$ of order $\sqrt{|H_1(M)|}$ such that $\tau(K, \chi) = 0$, for all characters $\chi$ which vanish on this subgroup. These are the characters which extend to $H_1(W)$, where $W$ is the 4-manifold obtained by taking the $k$-fold cyclic cover of the 4-ball branched over the slice disk. In other words $G = \ker(i_k: H_1(M) \to H_1(W))$, where $i$ denotes inclusion of $M$ into $W$.

If $A$ is a Seifert matrix for the knot $K$ corresponding to a certain basis for $H_1(F)$, then a presentation matrix for $H_1(M)$ is given by

$$\epsilon_k = [(A' - A)^{-1}A]^k - [(A' - A)^{-1}A]^k$$

with respect to a “lift” of a dual basis of $H_1(S^3 - F)$ (see [9, Section 1]). Here $A'$ denotes the transpose of $A$. The matrix $\epsilon_k$ defines a linear transformation from $H_1(F)$ to itself. In [9] P. Gilmer showed that for a slice knot $K$ the set of characters vanishing on $G$ can be identified with $(H \otimes \mathbb{Q}/\mathbb{Z}) \cap \ker(\epsilon_k \otimes \text{id}_{\mathbb{Q}/\mathbb{Z}})$, where $H$ is a metabolizer of the Seifert form. Clearly for equivariant slice knots we have the following.

**Corollary 4.3** Let $K$ be an equivariant slice knot. Then there exists an equivariant metabolizer $\overline{H}$ for $K$ such that the Casson–Gordon invariants
$\tau(K, \chi)$ vanish for all prime power order characters $\chi \in (\mathbb{H} \otimes \mathbb{Q}/\mathbb{Z}) \cap \ker(\varepsilon \otimes \text{id}_{\mathbb{Q}/\mathbb{Z}})$.

5 Branching Covers

The results in Theorem 4.2 and Corollary 4.3 are specific to the choice of an equivariant Seifert surface for the periodic knot, and therefore are not very useful for applications. In Theorem 5.2 we rephrase Corollary 4.3 in terms of homology groups of cyclic branched covers. In the following lemma we make a useful observation regarding the fixed points of the induced periodic action on the homology of branched covers. It can be proved using a transfer argument. (See [15, Proposition (2.5)].)

**Notation:** Given a prime $p$ and a group $G$ let $G_p$ denote the elements of $G$ with order a power of $p$.

**Lemma 5.1** Let $K$ be a period $q$ knot with the periodic map $f$ of $S^3$, let $M$ and $\overline{M}$ be the $k$-fold cyclic covers of $S^3$ branched over $K$ and $\overline{K}$, respectively, let $f_*$ denote the map on $H_1(M)$ induced by the lift of $f$, and let $p$ be a prime different from $q$, then

$$(\text{Fix}(f_*) \cap H_1(M)_p) \cong H_1(\overline{M})_p.$$ 

\[ \square \]

**Theorem 5.2** Let $K$ be a $q$-equivariant slice knot. Let $k$ be a power of a prime. Let $M$ be $k$-fold cyclic cover of $S^3$ branched over $K$ and let $\overline{M}$ be the $k$-fold cyclic cover of $S^3$ branched over the quotient knot. Let $p$ be a prime other than $q$. Then there exists a subgroup $G$ and an automorphism $f_*$ of $H_1(M)$ such that the following conditions are satisfied.
(i) $(\text{Fix}(f_\ast) \cap H_1(M)_p) \cong H_1(M)_p$,

(ii) $f_\ast(G) = G$,

(iii) $|G| = \sqrt{|H_1(M)|}$,

(iv) The linking form on $H_1(M)$ vanishes on $G$, and

(v) for a positive integer $r$ and a character $\chi : H_1(M) \to C_{2r}$, with $\chi|_G \equiv 1$, we have $\tau(K, \chi) = 0$.

**Proof:** Let $f_\ast$ be as in Lemma 5.1. Let $W$ is the 4-manifold obtained by taking the $k$-fold cyclic cover of the 4-ball branched over the equivariant slice disk. Then $M = \partial W$. Let $G$ denote the kernel of the map $\iota_* : H_1(M) \to H_1(W)$. Now (i) follows from Lemma 5.1, (ii) follows from Corollary 4.3, and (iii), (iv), (v) follow from [2, Theorem 2] and [9, Theorem 1].

We now use Theorem 5.2 to obtain conditions on Alexander polynomials of equivariant slice knots.

**Notation:** For distinct primes $p$ and $q$, let $f_q(p)$ denote the multiplicative order of $p \pmod{q}$, i.e., the least positive integer such that $p^{f_q(p)} \equiv 1 \pmod{q}$.

**Theorem 5.3** Let $K$ be a $q$-equivariant slice knot with $q$ a prime. Let $k$ be a power of some prime and $\zeta$ be a primitive $k$-th root of unity. Let $\Delta$ and $\overline{\Delta}$ be the Alexander polynomials of $K$ and its quotient knot, respectively. Let $p \neq q$ be a prime such that $p \mid \prod_{i=1}^k \Delta(\zeta_i^q)$ and $p \nmid \prod_{i=1}^k \overline{\Delta}(\zeta_i^q)$. Then $p^{f_q(p)} \mid \prod_{i=1}^k \Delta(\zeta_i^q)$.

**Proof:** (Compare [16, Theorem 5]. Note that in [16] $f_q(p)$ is defined to be the least positive integer such that $p^{f_q(p)} \equiv \pm 1 \pmod{q}$.) Let $M$, $\overline{M}$, $G$,
and \( f_s \) be as in Theorem 5.2. We have (see [10, Section 5]),

\[
|H_1(M)| = |\prod_{i=1}^{k} \Delta(\zeta^i)| \quad \text{and} \quad |H_1(M)| = |\prod_{i=1}^{k} \Xi(\zeta^i)| .
\]

Now, let \( G_p \) be the \( p \)-primary subgroup of \( G \). That is, \( G_p = G \cap H_1(M) \). Then \( f_s \) restricted to \( G_p \) is an automorphism, and since \( p \) does not divide \( |H_1(M)| \), by Theorem 5.3 (i) \( f_s \) has no fixed points in \( G_p \). It follows that \( G_p \) is a \( \mathbb{Z}[\zeta_q] \)-module. (See [15, Proposition 2.4].) Now by the characterization of \( \mathbb{Z}[\zeta_q] \) modules given [3, Theorem 2.5] we have nonnegative integers \( t, a_1, \ldots, a_t \) such that

\[
G_p \cong (C_p)^{a_1 f_0(p)} \oplus (C_{p^2})^{a_2 f_0(p)} \oplus \cdots \oplus (C_{p^t})^{a_t f_0(p)} .
\]

Here \( C_p \) denotes the cyclic group of order \( p \). Since \( |G_p| = \sqrt{|H_1(M)|} \), the result follows. \( \square \)

References


