1 Research Abstracts, Citations

- “Order in the concordance group and Heegaard Floer homology,” with S. Jabuka, submitted for publication, math.GT/0611023.

Abstract: We use the Heegaard-Floer homology correction terms defined by Ozsváth-Szabó to formulate a new obstruction for a knot to be of finite order in the smooth concordance group. This obstruction bears a formal resemblance to that of Casson and Gordon but is sensitive to the difference between the smooth versus topological category. As an application we obtain new lower bounds for the concordance order of small crossing knots.


Abstract: Let $S$ be either of the recently discovered knot concordance invariants of Ozsváth-Szabó or Rasmussen, suitably normalized to take value 1 on the positive trefoil knot. We prove: (1) If $D$ is the positive clasped $t$-twisted double of a knot $K$ and $D'$ is the negative clasped $t$-twisted double of $K$, then at most one of $S(D)$ and $S(D')$ can be nonzero, and (2) If $t < TB(K) + 1$, then $S(D) = 1$ and if $t > -TB(-K) - 1$ then $S(D) = 0$, where $TB$ denotes the Thurston-Bennequin number. A realization result is also proved: For any Seifert form $A$ of dimension $2g$ and for any integer $a$, $|a| < g + 1$, there is a knot $K$ with Seifert form $A$ and $S(K) = a$.

Cited in:


David Cimasoni, Slicing Bing doubles, math.GT/0609458.

Matthew Hedden, Knot Floer homology of Whitehead doubles, math.GT/0606094.

Matthew Hedden, Philip Ording, The Ozsváth-Szabó and Rasmussen concordance invariants are not equal, math.GT/0512348.

Matthew Hedden, Notions of positivity and the Ozsváth-Szabó concordance invariant, math.GT/0509499.

Charles Livingston, Slice knots with distinct Ozsváth-Szabó and Rasmussen Invariants, math.GT/0602631.

Ciprian Manolescu, Brendan Owens, A concordance invariant from the Floer homology of double branched covers, math.GT/0508065.

Abstract: Murasugi found two criteria that must be satisfied by the Alexander polynomial of a periodic knot. We generalize these to the case of twisted Alexander polynomials. Basic examples demonstrate the application of these new criteria. More delicate examples indicate their applicability to knots with trivial Alexander polynomial, including the two such knots with 11 crossings or less.

Cited in:

Daniel S. Silver, Susan G. Williams, Twisted Alexander Polynomials Detect the Unknot, math.GT/0604084.


Abstract: A knot with a cyclic period $q$ is called $q$-equivariant slice if the order $q$ periodic diffeomorphism of $S^3$ extends to that of $B^4$ and there is an invariant slice disk. This paper gives a characterization of Alexander polynomials of equivariant slice knots.

Cited in:

Jae Choon Cha, A characterization of the Murasugi polynomial of an equivariant slice knot, math.GT/0404403.


Abstract: In this paper we show that if $H_1(M_K) = \mathbb{Z}_p^n \oplus G$ with $p$ a prime congruent to 3 mod 4, $n$ odd, and $p$ not dividing the order of $G$, then $K$ is of infinite order in the concordance group $C_1$. We obtain several corollaries showing many of the twisted doubles, and unknotted number one knots with algebraic order 4 are, in fact, of infinite order in the concordance group.

Cited in:


Abstract: Two knots are S-equivalent if they are indistinguishable by Seifert matrices. We show that S-equivalence is generated by the doubled-delta move on knot diagrams. It follows as a corollary that a knot has trivial Alexander polynomial if and only if it can be undone by doubled-delta moves.
Cited in:


Carol Gwosdz Gee, Strong $S$-equivalence of ordered links, math.GT/0409440.


Abstract: Lickorish gave an elementary proof for the existence of the three-manifold invariant of Witten using a framed link description of the manifold and the formalization of the bracket polynomial as the Temperley-Lieb Algebra. R. Kaul determined a three-manifold invariant from link polynomials in $SU(2)$ Chern-Simons theory and commented that this invariant agreed with Witten’s partition function wherever computed but it was unknown whether the two are equivalent. We show that Kaul’s invariant is equivalent to Lickorish’s invariant by using representation theory of composite braiding in $SU(2)$ Chern-Simons theory.

Cited in:


Abstract: We prove that if the 2-fold cyclic cover of $S^3$ branched over a knot $K$ satisfies $|H_1(M_K)| = pm$ with $p$ a prime congruent to 3 mod 4 and gcd(p, m) = 1, then $K$ is of infinite order in the classical knot concordance group. As a corollary we show that if the Alexander polynomial of a knot satisfies $\Delta_K(t) = 5t^2 - 11t + 5$ then $K$ is of infinite order. Any higher dimensional knot with this polynomial is of order 4 in concordance.

Cited in:


Abstract: Two knots are said to be equivariant concordant if they cobound a periodic concordance. We give obstructions to equivariant concordance of classical knots in terms of the linking number of the knot with its axis, metabolizers of the Seifert form, and the Casson-Gordon invariants.

Cited in:


Abstract: For knots in $S^3$ we obtain new criteria for periodicity. First we obtain conditions in terms of the homology groups of cyclic branched covers of $S^3$ and the Alexander polynomial. Secondly we show that if a knot is periodic, then its Casson–Gordon invariants are preserved under the periodic action lifted to the abelian covers. As an application, we consider a family of knots with the Seifert form of a period 3 knot, and using Casson–Gordon invariants we show that knots in this family do not have period 3.

Cited in:


Abstract: For genus one knots we express the Casson-Gordon invariants in terms of the classical signatures, generalizing an earlier result of P. Gilmer. As an application it is shown that the pretzel knots $K(3, -5, 7)$ and $K(3, -5, 17)$ are not concordant to their reverses.
Cited in:


Abstract: In this paper we obtain criteria for periodicity of knots in $S^3$ in terms of the homology groups of cyclic branched covers of $S^3$ and the Alexander polynomial. We also study the relationship between the genus of a periodic knot and the Alexander polynomial. As an application we show that no eleven crossing knot has a period greater than 5.

Cited in:

Research Plans

The classical knot concordance group $C_1$ was first considered in work of Fox and Milnor [Fo,FM] in the 1960s. It plays an important role in low dimensional topology relating to both 3-dimensional and 4-dimensional manifolds.

A knot $K$ in $S^3$ is called slice if $(S^3, K) = \partial (B^4, D^2)$ where $D^2 \hookrightarrow B^4$ is a locally flat and properly embedded disk. We say $K_1$ is concordant to $K_2$ if $K_1 \# (-K_2)$ is slice. Here $-K$ is the orientation reversed mirror image of $K$. Concordance is an equivalence relation and the equivalence classes form an abelian group $C_1$ under the operation of connected sum.

$C_1$ is known to be an infinitely generated, countable abelian group. Besides classes of order infinity it also contains classes of order two represented by negative amphicheiral knots. Nothing is known about torsion in this group beyond this.

To contrast, in higher dimensions, it is known that the analogous groups $C_{2n}$ are all trivial, and $C_{2n+1}$ for $n \geq 1$, is isomorphic to the algebraic concordance group of Witt classes of Seifert forms $\mathbb{Z}^\infty \oplus \mathbb{Z}_2^\infty \oplus \mathbb{Z}_4^\infty$ defined by J. Levine [L1,L2]. It is also known that there are order two classes in higher odd dimensions which are not represented by negative amphicheiral knots [CM]. Analogous questions in dimension three remain open.

In dimension three Levine in [L1,L2] showed that there is a homomorphism from $C_1$ onto the algebraic concordance group, and Casson and Gordon in [CG1,CG2] showed that the kernel of this map is non-trivial.

Requiring the slicing 2-disk to be smooth, and not just locally flat, takes us into the world of smooth concordance.

By the work of Freedman [Fr, FQ] we know that knots with Alexander polynomial one are topologically slice. Using gauge theory [FS] many of these knots are shown to be not smoothly slice. Further examples are given using invariants $\tau$ and $s$ obtained from Heegaard-Floer and Khovanov homology, respectively.

I have several on-going projects related to knot concordance. I will describe some of these in the subsequent sections.

References


Research Plans


2.1 Torsion in the knot concordance group

In 1977 Gordon [G] (see also [K] problems 1.32 and 1.94) asked whether every order 2 class in $C_1$ is represented by an amphicheiral knot. The knot concordance group may be very simple in that every knot is either slice, or infinite order, or isotopic to its concordance inverse. However, no progress has been made in this regard.

Questions in this section relate to torsion in the concordance group. These are difficult questions and any partial progress would be desirable.

In joint work with Jabuka [JN] we have used the Heegaard-Floer correction terms to obtain new obstructions for a knot to be of smooth finite order in $C_1$. We use the fact, that if a knot $K$ in $S^3$ is of order $n$ in $C_1$ then the $n$-fold connected sum of the double cover of $S^3$ branched along $K$, with itself, bounds a rational homology 4-ball. This is reflected in the vanishing of certain sums of correction terms. Using these observations we ruled out (smooth) concordance order 4 for several prime knots with ten or fewer crossings.

In joint work with Livingston [LN1, LN2] we used Casson-Gordon invariants to show that if the $p$-primary subgroup of $H_1$ of the double cover of $S^3$ branched over $K$ is a cyclic group $\mathbb{Z}_p^n$, with $p$ a prime congruent to 3 mod 4 and $n$ odd, then $K$ represents an infinite order class in $C_1$.

In [COT1, COT2, CT] a geometric filtration

$$0 \subset \cdots \subset F_{n,5} \subset F_{n,0} \subset \cdots \subset F_{1,5} \subset F_{1,0} \subset F_{0,5} \subset C_1$$

of the concordance group was introduced and studied by Cochran-Orr-Teichner.

It was shown that the quotient groups $F_{n,5}/F_{n,0}$ are nontrivial at each stage, $C/F_{1,0}$ is the algebraic concordance group, and Casson-Gordon invariants vanish for elements of $F_{1,5}$.

It is unknown whether the Casson-Gordon invariants, even though entirely captured by the bottom parts in this filtration, are, in fact, sufficient to eliminate order four and above in $C_1$. It would be interesting to explore whether examples can be constructed for which the Casson-Gordon obstruction for finite order vanishes, but higher order $L^2$ signatures provide it.

**Problem 1** Can we find higher order obstructions to torsion in the the COT filtration of $C_1$?
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The Jabuka-Naik results apply to individual knots with small number of crossings, and the Livingston-Naik results obstruct S-equivalence classes of finite algebraic order from being finite order in $C_1$. To show that Levine’s homomorphism onto the algebraic concordance group does not split, one needs to find a finite order algebraic concordance class which does not have a concordance representative of the same order.

Continuing the Livingston-Naik scheme to find such a class, we need to consider the case that the $p$-primary subgroup of $H_1$ of the double branched cover is a nontrivial direct sum of cyclic groups. We have successfully dealt with a number of special cases of direct sums, however, the current techniques do not seem to be sufficient to obtain the full extension.

One can, however, ask a possibly easier question, which Livingston and I are currently exploring.

**Problem 2** If $K$ satisfies the hypothesis in the Livingston-Naik Theorem above, and $J$ is an algebraic slice knot, is it possible for $K \# J$ to have finite order in $C_1$?

Invariants, such as the Tristram-Levine signatures (topological), Ozsváth-Szabó invariant $\tau$ (smooth), and the Rasmussen invariant $s$ (smooth), each being a homomorphism to $\mathbb{Z}$, vanish on finite concordance order knots.

**References**


2.2 Link Concordance

It was shown recently by Davis [D] that any two-component link with Alexander polynomial one is topologically concordant to the Hopf link. Using tools which distinguish between smooth and topological concordance, Chris Herald, Slaven Jabuka and myself are considering the following.

Problem 3 Obtain obstructions to an Alexander polynomial one link being smoothly concordant to the Hopf link using the Heegaard-Floer homology.

Using Alexander polynomial one knots which are not smoothly slice, for example the \((-3, 5, 7)\) pretzel knot, it is easy to obtain examples of Alexander polynomial one links which are not smoothly concordant to the Hopf link. One hopes that the Heegaard-Floer obstruction will help make progress on the following problem posed in [D].

Problem 4 Is there a two-component link with Alexander polynomial one which is not smoothly concordant to the Hopf link, but each of whose components are smoothly concordant to the unknot?

REFERENCES


2.3 Equivariant Concordance

A knot \(K \subset S^3\) is called periodic, of period \(q\), if there is an order \(q\) transformation of \(S^3\) that leaves \(K\) invariant and has as fixed point set a circle \(A\) disjoint from \(K\). The axis \(A\) is always unknotted, equivalently, the quotient of \(S^3\) by the \(\mathbb{Z}_q\)-action is \(S^3\), and the image of \(K\) is called a quotient knot.

If a periodic knot is slice, one can ask whether it is equivariant slice, that is, whether the order \(q\) transformation extends to the 4-ball leaving a slice disk invariant. We studied (topological) equivariant concordance in [DN,N].

In [N] it was shown that an equivariant slice knot has linking number one with the axis of rotation, and the knot with its axis is concordant to the Hopf link. Conversely, a \(q\)-equivariant slice knot can be formed by starting with a 2-component link with linking number one, which is concordant to the Hopf link, one of whose components is unknotted, and then taking the \(q\)-fold cover of \(S^3\) branched over the unknotted component. This motivates the following questions. An answer to Problem 3 may provide insight to Problem 6.

Problem 5 Obtain obstructions for a link with linking number one, and one of the components unknotted, to be concordant to the Hopf link.

Problem 6 Obtain obstructions to smooth equivariant sliceness.

With J. F. Davis, we are currently investigating the following question.
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Problem 7 Must a periodic knot with Alexander polynomial one be (topologically) equivariant slice?

The proof in [D] that any Alexander polynomial one link is concordant to the Hopf link involved choosing the surgery problem so that an obstruction given in terms of the signature vanishes. The corresponding obstruction in the equivariant slice case is in terms of G-signatures which will have to simultaneously vanish.

REFERENCES


2.4 Twisted Alexander polynomials

Twisted Alexander polynomials are generators of order ideals of the Alexander module of a knot, twisted by a representation of the knot group. For suitable representations these are related to determinants of the Casson-Gordon invariants. See [KL]. If the knot is slice, they satisfy a factorization condition similar to the usual Alexander polynomial.

In [HLN] Hillman, Livingston, and myself obtained obstructions for periodicity in terms of twisted polynomials corresponding to a representation $\rho$ of $GL_n(\mathbb{Q})$. We proved that

$$\Delta_{K,\rho}(t) = \Delta_{\bar{K},\bar{\rho}}(t) \prod_{i=1}^{q-1} F(t, \zeta_i^q),$$

where $F(t, s) \in R[t^\pm, s^\pm]$ is a related twisted polynomial of the two-component link consisting of the quotient knot and its axis and $\zeta_q = e^{2\pi i}$. Davis and I showed in [DN], that for a (topologically) equivariant slice knot, the two variable polynomial occurring in the analogous Murasugi factorization of the Alexander polynomial factors as $p(t, s) \cdot p(t^{-1}, s^{-1})$. It is natural to ask whether $F(t, s)$ occurring in the factorization of a twisted polynomial would factor similarly. Davis and I are currently working on this.

Problem 8 Generalize Theorem 1.9 in [DN] to provide obstructions to equivariant sliceness in terms of twisted Alexander polynomials.

If the answer to Problem 7 is negative, it may provide Alexander polynomial one candidates to check against the more subtle obstructions to equivariant sliceness given by the twisted polynomials.

REFERENCES

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2.5 Other problems: Heegaard-Floer homology

Our work in [JN, LN] relates to new advances in knot theory and in 3- and 4-dimensional manifolds coming from Heegaard-Floer homology and Khovanov homology. In [JN] we use Heegaard-Floer homology to address interesting questions about concordance. This is a rich area which provides many new tools. These methods also bring out the subtle differences between topological and smooth concordance.

Chris Herald, myself and Stanislav Jabuka have recently submitted joint NSF Proposal 6634191, entitled “The Knot Concordance Group and Heegaard-Floer Homology,” describing several problems that we propose to solve together using our combined expertise in gauge theory, representations of knot groups, Heegaard Floer homology, knot concordance and knot periodicity.

We intend to use existing tools and also develop new ones, such as an extension of the Manolescu-Owens $\delta$-invariant to $2^n$-fold branched covers.

REFERENCES
