This handout discusses the Envelope Theorem and its application to the profit function.

**Part A: Envelope Theorem (Unconstrained Case)**

Consider the following vectors of \( x = (x_1, x_2, \ldots, x_n) \) and \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m) \). Let \( F(x, \alpha) \) be a continuous and differentiable function. We recognize that function \( F \) will have a different optimizing solution of \( x \) for each \( \alpha \). In other words, there will be an envelope of optimizing solution of \( x \) determined for each \( \alpha \), i.e., \( x = x^*(\alpha) \).

Applying the first-order condition (FOC) to \( F(x, \alpha) \), we get:

\[
\frac{\partial F(x, \alpha)}{\partial x_i} = 0 \quad \forall i = 1, \ldots, n. \tag{1}
\]

The optimum solution of \( x \), i.e., \( x = x(\alpha) = [x_1(\alpha), x_2(\alpha), \ldots, x_n(\alpha)] \) is obtained by solving Equation (1), provided that the sufficient second-order condition for optimization is satisfied.

Let \( V(\alpha) \) be the value function defined as:

\[
V(\alpha) = F(x(\alpha), \alpha) \tag{2}
\]

The Envelope Theorem states that:

\[
\frac{\partial V(\alpha)}{\partial \alpha_j} = \frac{\partial F(x, \alpha)}{\partial \alpha_j} \bigg|_{x = x(\alpha)} ; \forall j = 1, \ldots, m. \tag{3}
\]

The Envelope Theorem is a short-cut to derive optimizing solution of \( x \) determined by \( \alpha \).

**Part B: Example**

Consider \( F(x, \alpha) = \ln(x) - \alpha x \). The first and second order conditions are:

FOC: \( \frac{\partial F}{\partial x} = \frac{1}{x} - \alpha = 0; \)

or \( x^* = \frac{1}{\alpha} \).
SOC: \[
\frac{\partial}{\partial x} \left( \frac{\partial F}{\partial x} \right) = \frac{\partial^2 F}{\partial x^2} = -\frac{1}{x^2} < 0.
\]

So the solution of \( x^* = \frac{1}{\alpha} \) is a maximum.

The value function is:

\[
V(\alpha) = \ln \left( \frac{1}{\alpha} \right) - \alpha \left( \frac{1}{\alpha} \right) = -\ln(\alpha) - 1.
\]  

(4)

Taking the derivative of the value function in Equation (4) with respect to \( \alpha \) yields:

\[
\frac{dV(\alpha)}{d\alpha} = -\frac{1}{\alpha}.
\]

(5)

Applying the Envelope Theorem to \( F(x,\alpha) = \ln(x) - \alpha x \) yields:

\[
\frac{\partial F}{\partial \alpha} \bigg|_{x=\frac{1}{\alpha}} = -x \bigg|_{x=\frac{1}{\alpha}} = -\frac{1}{\alpha}.
\]

(6)

The use of the Envelope Theorem allows us to skip the extra step of deriving the value function.

**PART C: Application of the Envelope Theorem to Profit Maximization**

Consider a strictly concave production function for a two-input case, \( y = f(x_1, x_2) \). The objective is to maximize:

\[
\pi = pf(x_1, x_2) - w_1 x_1 - w_2 x_2
\]

(7)

As discussed in class, applying the FOC to Equation (7) yields:

\[
 pf_1 = w_1
\]

(8)

\[
 pf_2 = w_2
\]

(9)

The sufficient SOC condition for profit-maximization is achieved by the requirement that the profit function is strictly concave. The input demand functions, \( x_1 = x_1(w_1, w_2, \pi) \)
and $x_2 = x_2(w_1, w_2, p)$, can be derived by solving Equations (8) and (9). Substituting the factor demand function in Equation (7) yields:

$$\pi = pf[x_1(w_1, w_2, p), x_2(w_1, w_2, p)] - w_1x_1(w_1, w_2, p) - w_2x_2(w_1, w_2, p)$$  \hspace{1cm} (10)

We recognize that the profit function in Equation (10) changes as a result of a change in input and commodity prices. Applying the Envelope Theorem to Equation (10) yields:

$$\frac{\partial \pi}{\partial w_1} = -x_1(w_1, w_2, p)$$  \hspace{1cm} (11)

$$\frac{\partial \pi}{\partial w_2} = -x_2(w_1, w_2, p)$$  \hspace{1cm} (12)

$$\frac{\partial \pi}{\partial p} = y(w_1, w_2, p)$$  \hspace{1cm} (13)

The results in Equations (11)-(13) are the presentation of the duality theorem applied to the profit function. The results in Equations (11)-(13) can be proven by differentiating the profit function in Equation (10) with respect to $w_1$, $w_2$ or $p$ and apply the results from FOC in Equations (8) and (9).

Let’s differentiate the profit function in Equation (10) with respect to $w_1$. We get:

$$\frac{\partial \pi}{\partial w_1} = pf_1 \frac{\partial x_1}{\partial w_1} + pf_2 \frac{\partial x_2}{\partial w_1} - x_1 - w_1 \frac{\partial x_1}{\partial w_1} - w_2 \frac{\partial x_2}{\partial w_1}$$  \hspace{1cm} (14)

Rearranging the terms in the right-hand-side of Equation (14), we get:

$$\frac{\partial \pi}{\partial w_1} = (pf_1 - w_1) \frac{\partial x_1}{\partial w_1} + (pf_2 - w_2) \frac{\partial x_2}{\partial w_1} - x_1$$  \hspace{1cm} (15)

Applying Equations (8) and (9) to Equation (15) yields the result in Equation (11), i.e.,

$$\frac{\partial \pi}{\partial w_1} = -x_1(w_1, w_2, p)$$, which is the application of the Envelope Theorem to the profit function in Equation (10).