1 Functions

**Definition 1**  A function \( f \) from (or on) a set \( X \) to (or into) a set \( Y \) is a rule that assigns to each \( x \in X \) a unique element \( f(x) \) in \( Y \).

**Definition 2**  The collection \( G \) of pairs \((x, f(x))\) in \( X \times Y \) is called the graph of the function \( f \).

The word “mapping” is often used for “function”.

We will express the fact that \( f \) is a function from \( X \) into \( Y \) by writing

\[ f : X \to Y. \]

**Definition 3**  The set \( X \) is called the domain of \( f \). The set of values taken by \( f \), that is, the set

\[ \{y \in Y : \exists x \text{ such that } y = f(x)\} \]

is called the range of \( f \).

**Definition 4**  For \( A \subseteq X \), the image \( f(A) \) of \( A \) under \( f \) is the set of elements of \( Y \) such that \( y = f(x) \) for some \( x \in A \):

\[ f(A) = \{y \in Y : \exists x \in A \text{ such that } y = f(x)\}. \]
Definition 5 Function $f$ is onto $Y$ if and only if $Y = f(X)$.

Definition 6 For $B \subseteq Y$, the inverse image $f^{-1}(B)$ of $B$ is the set of those elements $x \in X$ for which $f(x) \in B$:

$$f^{-1}(B) = \{x \in X : f(x) \in B\}.$$

Definition 7 Function $f$ is called one-to-one (or univalent, or injective) if and only if $f(x_1) = f(x_2)$ only if $x_1 = x_2$.

Definition 8 Functions $f : X \to Y$ which are one-to-one and onto are called one-to-one correspondences (or bijective).

Definition 9 If function $f$ is a one-to-one correspondence between $X$ and $Y$ then there exists function $g : Y \to X$ such that $g(f(x)) = x$ and $f(g(y)) = y$. The function $g$ is called the inverse of $f$ and is frequently denoted by $f^{-1}$.

Definition 10 Let $f : X \to Y$ and $g : Y \to Z$. The function $h : X \to Z$ defined as $h(x) = g(f(x))$ for all $x \in X$ is called the composition of $g$ with $f$ and is denoted by $g \circ f$.

Definition 11 Let $f : X \to Y$ and $A \subseteq X$. The function $g : A \to Y$ defined as $g(x) = f(x)$ for all $x \in A$ is called the restriction of function $f$ to $A$ and is sometimes denoted by $f|A$.

Exercise Set 1

Exercise 1: Let $f : X \to Y$. Show that $f$ is onto if there is a mapping $g : Y \to X$ such that $f \circ g$ is the identity map on $Y$, that is, $f(g(y)) = y$ for all $y \in Y$.

Exercise 2: Show that:

(i) $f \left( \bigcup_{\lambda \in \Lambda} A_{\lambda} \right) = \bigcup_{\lambda \in \Lambda} f[A_{\lambda}]$;

(ii) $f \left( \bigcap_{\lambda \in \Lambda} A_{\lambda} \right) \subseteq \bigcap_{\lambda \in \Lambda} f[A_{\lambda}]$;
(iii) Provide an example where \( f \left[ \bigcap_{\lambda \in \Lambda} A_\lambda \right] \neq \bigcap_{\lambda \in \Lambda} f[A_\lambda] \)

**Exercise 3:** Show that:

(i) \( f^{-1} \left[ \bigcup_{\lambda \in \Lambda} A_\lambda \right] = \bigcup_{\lambda \in \Lambda} f^{-1}[A_\lambda] \)

(ii) \( f^{-1} \left[ \bigcap_{\lambda \in \Lambda} A_\lambda \right] = \bigcap_{\lambda \in \Lambda} f^{-1}[A_\lambda] \)

## 2 Metric Spaces

**Definition 12** A **metric** on a nonempty set \( X \) is a function \( d: X \times X \to \mathbb{R}_+ \) satisfying the following properties:

a) **Non-negativity:** For all \( x, y \in X \), \( d(x, y) \geq 0 \).

b) **Discrimination:**

\[
d(x, x) = 0 \text{ and }\]
\[
d(x, y) = 0 \rightarrow x = y.
\]

c) **Symmetry:** For all \( x, y \in X \), \( d(x, y) = d(y, x) \).

d) **Triangle Inequality:** For all \( x, y, z \in X \), \( d(x, z) \leq d(x, y) + d(y, z) \).

**Definition 13** The pair \( (X, d) \), where \( d \) is a metric on \( X \), is called a **metric space**.

**Example 14** The natural metric on \( \mathbb{R} \) is

\[
d(x, y) = |x - y|.
\]

There are several natural metrics on \( \mathbb{R}^M \). The **Euclidean metric** is defined by

\[
d(x, y) = \sqrt{\sum_{i=1}^{M} (x_i - y_i)^2}.
\]

The \( l_1 \) metric (also called the taxi-cab metric for \( \mathbb{R}^2 \)) is defined by

\[
d(x, y) = \sum_{i=1}^{M} |x_i - y_i|.
\]
The sup metric or uniform metric is defined by
\[ d(x, y) = \max_{i=1,...,M} |x_i - y_i|. \]

Example 15 Let \( X \) be any set and define metric \( d \) by \( d(x, x) = 0 \) and \( d(x, y) = 1 \) if \( x \neq y \). Then, \( d \) is a metric on \( X \) called the discrete metric.

Exercise Set 2

Exercise 4: Demonstrate that the metrics in the previous two examples satisfy all of the four properties of a metric.

3 Open and closed sets

Definition 16 An (open) \( \varepsilon \)-ball about \( x \in E \), where \( E \) is a set in a metric space \( (X, d) \), is defined as
\[ B_\varepsilon(x) = \{ y \in E : d(y, x) < \varepsilon \}. \]

Definition 17 A set \( E \) in a metric space \( (X, d) \) is open iff for each \( x \in E \) there exists \( \varepsilon > 0 \) such that \( B_\varepsilon(x) \subseteq E \). A set is closed if its complement is open.

Definition 18 A point \( x \in X \) is called a point of closure of set \( E \) in a metric space \( (X, d) \) if \( \forall \delta > 0 \exists y \in E \) such that \( d(x, y) < \delta \). The set of points of closure of \( E \) is denoted by \( \overline{E} \) (or cl(\( E \))).

The closure of a set \( E \) consists of all points which are intuitively “close to \( E \”).

Theorem 19 A set \( E \) is closed if and only if \( E = \overline{E} \).
Frequently, condition $E = \overline{E}$ is used as a definition of a closed set.

The definition of a point of closure is closely related to the definition of a limit point:

**Definition 20** A point $x \in X$ is called a limit point of set $E$ in a metric space $(X, d)$ if $\forall \delta > 0 \ \exists y \in E$ such that $y \neq x$ and $d(x, y) < \delta$.

The difference between the two definitions is subtle but important — namely, in the definition of the limit point, every open ball about point $x$ must contain a point of the set other than $x$ itself.

**Theorem 21** Every limit point is a point of closure, but not every point of closure is a limit point.

**Definition 22** A point of closure which is not a limit point is called an isolated point.

**Theorem 23** The closure of a set $E$ is equal to the union of $E$ and the set of limit points of $E$.

**Example 24** In the natural metric $d(x, y) = |x - y|$ on $\mathbb{R}$, the $\varepsilon$-ball about $x \in X$ is just the interval $(x - \varepsilon, x + \varepsilon)$. This makes it clear that each interval of the form $(a, b)$ is an open set. There are obviously open sets of forms other than $(a, b)$ (Which are these forms?).

**Theorem 25** The open sets in a metric space $(X, d)$ have the following properties:

a) Any union of open sets is open;
b) Any finite intersection of open sets is open;
c) $\emptyset$ and $X$ are open.
Given the definition of a closed set, the corresponding theorem for the closed sets is:

**Theorem 26** The closed sets in a metric space \((X, d)\) have the following properties:

- a) Any intersection of closed sets is closed;
- b) Any finite union of closed sets is closed;
- c) \(\emptyset\) and \(X\) are closed.

**Definition 27** The **interior** \(\text{Int}(E)\) of set \(E\) in a metric space \((X, d)\) is the open set

\[
\text{Int}(E) = \{x \in E : \exists \varepsilon > 0 \text{ such that } d(y, x) < \varepsilon \text{ and } y \in X \text{ imply } y \in E\}.
\]

**Definition 28** The **boundary** \(B\text{dry}(E)\) (also denoted \(\partial E\)) of set \(E\) in a metric space \((X, d)\) is the closed set

\[
B\text{dry}(E) = \overline{E} \setminus \text{Int}(E).
\]

Every point of a set is either an interior point or a boundary point.

**Definition 29** A set \(E\) in a metric space \((X, d)\) is **bounded** iff \(E \subseteq B_M(x)\) for some \(x \in E\) and \(0 < M < +\infty\).

**Definition 30** A set \(E\) in the Euclidean space \(\mathbb{R}^M\) is **compact** iff it is closed and bounded.

**Exercise Set 3**

**Exercise 5:** Prove Theorem 19.

**Exercise 6:** Prove Theorem 21.

**Exercise 7:** Prove Theorem 23.

**Exercise 8:** Prove Theorem 25.

**Exercise 9:** Prove that the interior of any set is open.

**Exercise 10:** Prove that the boundary of any set is closed.

**Exercise 11:** Prove that \(\text{Int}(E) = (\overline{E})^c\).
4 Infimum and Supremum

Definition 31 The greatest lower bound (or infimum, or \inf) of a subset \( S \) of a partially ordered set \((P, \leq)\), denoted as \( \inf(S) \), is an element \( l \) of \( P \) such that
1. \( l \leq x \) for all \( x \in S \), and
2. For any \( p \in P \) such that \( p \leq x \) for all \( x \in S \) it holds that \( p \leq l \).

Definition 32 The least upper bound (or supremum, or \sup) of a subset \( S \) of a partially ordered set \((P, \leq)\), denoted as \( \sup(S) \), is an element \( u \) of \( P \) such that
1. \( u \geq x \) for all \( x \in S \), and
2. For any \( p \in P \) such that \( p \geq x \) for all \( x \in S \) it holds that \( p \geq u \).

Definition 33 (real numbers \( \mathbb{R} \)) The infimum of a set \( S \subseteq \mathbb{R} \) is \( l \in \mathbb{R} \) such that
1. \( l \leq x \) for all \( x \in S \), and
2. \( \forall \varepsilon > 0 \ \exists x \in S \) such that \( l + \varepsilon > x \).

Definition 34 (real numbers \( \mathbb{R} \)) The supremum of a set \( S \subseteq \mathbb{R} \) is \( u \in \mathbb{R} \) such that
1. \( u \geq x \) for all \( x \in S \), and
2. \( \forall \varepsilon > 0 \ \exists x \in S \) such that \( u - \varepsilon < x \).

5 Sequences

Definition 35 By a sequence \( \{x_n\}_{n=1}^{\infty} \) from a nonempty set \( X \), we mean an ordered set of the form \((x_1, x_2, ...)\) where \( x_i \in X \) for all \( i \in \{1, 2, ..., \infty\} \). More formally, a sequence \( \{x_n\}_{n=1}^{\infty} \) is a function from the set of natural numbers \( N \) into \( X \). That is, sequence \( \{x_n\}_{n=1}^{\infty} \) is the function \( f : N \to X \) such that \( f(n) = x_n \) for all \( n \in N \).

Definition 36 Consider a sequence \( \{x_n\}_{n=1}^{\infty} \) and a strictly increasing function \( \sigma : N \to N \) (that is, \( \sigma(k) < \sigma(l) \) for any \( k, l \in N \) with \( k < l \)). The sequence \( \{x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, ...\} = \{x_{\sigma(k)}\}_{k=1}^{\infty} \) is called a subsequence of sequence \( \{x_n\}_{n=1}^{\infty} \).
Example 37 Consider sequence $(0, 1, 0, 1, 0, 1, \ldots)$. The subsequence formed by taking the odd-numbered elements of the sequence is given by $(0, 0, 0, \ldots)$. The subsequence formed by taking the even-numbered elements of the sequence is given by $(1, 1, 1, \ldots)$.

Definition 38 Let $(X, d)$ be a metric space. A sequence $\{x_n\}_{n=1}^{\infty}$ converges to $x \in X$, written $\lim_{n \to \infty} x_n = x$, iff $d(x_n, x)$ converges to 0 as a sequence of real numbers, i.e. $\lim_{n \to \infty} d(x_n, x) = 0$. Equivalently, $\lim_{n \to \infty} x_n = x$ iff 

$$\forall \varepsilon > 0 \exists N \forall n \geq N \ d(x_n, x) < \varepsilon.$$ 

Definition 39 (Real numbers $R$) A sequence of real numbers $\{x_n\}_{n=1}^{\infty}$ converges to $x \in R$, written $\lim_{n \to \infty} x_n = x$, iff $|x_n - x|$ converges to 0 as a sequence of real numbers, i.e. $\lim_{n \to \infty} |x_n - x| = 0$. Equivalently, $\lim_{n \to \infty} x_n = x$ iff

$$\forall \varepsilon > 0 \exists N \forall n \geq N \ |x_n - x| < \varepsilon.$$

Definition 40 ($\mathbb{R}^M$ with the Euclidean metric) A sequence $\{\mathbf{x}_n\}_{n=1}^{\infty}$ from $\mathbb{R}^M$ (where $\mathbf{x}_n = (x_{1n}, x_{2n}, \ldots, x_{Mn})$) converges to $\mathbf{x} = (x_1^1, x_2^2, \ldots, x_M^M) \in \mathbb{R}^M$, written $\lim_{n \to \infty} \mathbf{x}_n = \mathbf{x}$, iff $\sqrt{\sum_{i=1}^{M} (x_i^i - x_i^i)^2}$ converges to 0 as a sequence of real numbers, i.e. $\lim_{n \to \infty} \sqrt{\sum_{i=1}^{M} (x_i^i - x_i^i)^2} = 0$. Equivalently, $\lim_{n \to \infty} \mathbf{x}_n = \mathbf{x}$ iff

$$\forall \varepsilon > 0 \exists N \forall n \geq N \ \sqrt{\sum_{i=1}^{M} (x_i^i - x_i^i)^2} < \varepsilon.$$

6 Continuous Functions

Definition 41 If $(X, \rho)$ and $(Y, \sigma)$ are metric spaces, a function $f : X \to Y$ is continuous at $x_0 \in X$ iff for each sequence $\{x_n\}_{n=1}^{\infty}$ that converges to $x_0$, $\lim_{n \to \infty} f(x_n) = f(x_0)$. Equivalently, $f : X \to Y$ is continuous at $x_0 \in X$ iff

$$\forall \varepsilon > 0 \exists \delta > 0 \ \forall x \in X \ [\rho(x, x_0) < \delta \rightarrow \sigma(f(x), f(x_0)) < \varepsilon].$$
Definition 42 (Continuity for a real-valued function on $\mathbb{R}$ with the natural metric on $\mathbb{R}$) Let $f : X \rightarrow Y$, where $X, Y \subseteq \mathbb{R}$. $f : X \rightarrow Y$ is continuous at $x_0 \in X$ iff for each sequence $\{x_n\}_{n=1}^{\infty}$ that converges to $x_0$, $\lim_{n \to \infty} f(x_n) = f(x_0)$. Equivalently, $f : X \rightarrow Y$ is continuous at $x_0 \in X$ iff

$$\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall x \in X \quad ||x - x_0|| < \delta \rightarrow |f(x) - f(x_0)| < \varepsilon.$$ 

We can rephrase the definition of a continuous function between metric spaces in terms of open sets in these spaces, thus avoiding explicit mention of the metrics involved:

Theorem 43 If $(X, \rho)$ and $(Y, \sigma)$ are metric spaces, a function $f : X \rightarrow Y$ is continuous at $x_0 \in X$ iff for each open set $V$ in $Y$ containing $f(x_0)$ there is an open set $U$ in $X$ containing $x_0$ such that $f(U) \subseteq V$.

Thus, we can carry the definition of a continuous function to any setting where we can carry a “reasonable” notion of an open set. A “reasonable” definition of an open set is introduced in the following section, where we introduce a mathematical construct that includes metric spaces as a special case.

7 Topological Spaces

Definition 44 A topology on a set $X$ is a collection $\tau$ of subsets of $X$, called the open sets, satisfying:

a) Any union of elements of $\tau$ belongs to $\tau$;

b) Any finite intersection of elements of $\tau$ belongs to $\tau$;

c) $\emptyset$ and $X$ belong to $\tau$.

Definition 45 The pair $(X, \tau)$, where $\tau$ is a topology on $X$, is called a topological space.

Definition 46 Complements of open sets are called closed.
Definition 47 Let \((X, d)\) be a metric space. Then, by Theorem 25, the open sets defined by Definition 17 form a topology called the **metric topology** \(\tau_d\). Whenever \((X, \tau)\) is a topological space whose topology \(\tau\) is the metric topology \(\tau_d\) for some metric \(d\) on \(X\), we call \((X, d)\) a **metrizable topological space**. Every metric space \((X, d)\) defines a metrizable topological space \((X, \tau_d)\) and given a metrizable topological space \((X, \tau)\), one can always find many metrics \(d\) on \(X\) such that \(\tau_d = \tau\). Two metrics generating the same topology are **equivalent**.

The Euclidean, \(l_1\), and sup metrics on \(\mathbb{R}^n\) are equivalent.

Definition 48 A property of a metric space that can be expressed in terms of open sets without mentioning a specific metric is called a **topological property**.

There are many topological spaces that are not metrizable.

One can easily introduce a definition of convergence of a sequence in a topological space as well as other notions that you are familiar with for metric spaces. We only provide a definition of continuity of a function:

Definition 49 If \((X, \tau)\) and \((Y, \upsilon)\) are topological spaces, a function \(f : X \to Y\) is **continuous** at \(x_0 \in X\) iff for each open set \(V\) in \(Y\) containing \(f(x_0)\) there is an open set \(U\) in \(X\) containing \(x_0\) such that \(f(U) \subseteq V\).

Note that this definition is similar to the one given for metric spaces.

8 Lower and Upper Semicontinuity, and Continuity

Definition 50 A real-valued function \(f : X \to \mathbb{R}\) is **upper semicontinuous** if for each \(a \in \mathbb{R}\), the upper contour set \(\{x : f(x) \geq a\}\) is closed (or equivalently \(\{x : f(x) > a\}\) is open). It is **lower semicontinuous** if every lower contour set \(\{x : f(x) \leq a\}\) is closed (or equivalently \(\{x : f(x) < a\}\) is open).
The above definition is valid when $X$ is a general (not necessarily metrizable) topological space.

**Theorem 51** A real-valued function is **continuous** if and only if it is both upper and lower semicontinuous.

Note that $f$ is upper semicontinuous if and only if $-f$ is lower semicontinuous. We can even talk about semicontinuity at a point. We define it for the case when $X$ is a metric space (One can easily introduce a similar definition for a general topological space.)

**Definition 52** Let $(X, d)$ be a metric space. The real-valued function $f : X \to \mathbb{R}$ is **upper semicontinuous** at $x \in X$ iff

$$f(x) \geq \lim_{y \to x} \sup \{ f(y) : \inf_{\varepsilon > 0} \sup_{0 < d(x, y) < \varepsilon} f(y) \}.$$ 

Similarly, $f$ is **lower semicontinuous** at $x \in X$ iff

$$f(x) \leq \lim_{y \to x} \inf \{ f(y) : \sup_{\varepsilon > 0} \inf_{0 < d(x, y) < \varepsilon} f(y) \}.$$ 

Equivalently, $f$ is **upper semicontinuous** at $x \in X$ iff

$$\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall x \in X \ [d(x, y) < \delta \Rightarrow f(y) < f(x) + \varepsilon].$$

Similarly, $f$ is **lower semicontinuous** at $x \in X$ iff

$$\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall x \in X \ [d(x, y) < \delta \Rightarrow f(y) > f(x) - \varepsilon].$$

9 **Correspondences, Lower and Upper Hemicontinuity**

Let $X$ and $Y$ be topological spaces. If it is hard for you to think of $X$ and $Y$ as topological spaces, think of them as metric spaces or a specific metric space (for example, the Euclidean space).
Definition 53 A correspondence \( \varphi : X \rightrightarrows Y \) associates to each point in \( X \) a subset of \( Y \). For a correspondence \( \varphi \), let

\[
\text{Gr}(\varphi) = \{(x, y) \in X \times Y : x \in X, y = \varphi(x)\}
\]
denote the graph of \( \varphi \).

Definition 54 The upper (or strong) inverse of \( E \) under \( \varphi \) is defined by

\[
\varphi^U(E) = \{x \in X : \varphi(x) \subseteq E\}.
\]

The lower (or weak) inverse of \( E \) under \( \varphi \) is defined by

\[
\varphi^L(E) = \{x \in X : \varphi(x) \cap E \neq \emptyset\}.
\]

Definition 55 A correspondence \( \varphi \) is upper hemicontinuous (uhc) at \( x \) if whenever \( x \) is in the upper inverse of an open set, so is a neighborhood of \( x \). That is, \( \varphi \) is uhc at \( x \) if \( \forall \) open set \( E \) with \( x \in \varphi^U(E) \) \( \exists \) open set \( A \) such that \( x \in A \) and \( A \subseteq \varphi^U(E) \).

\( \varphi \) is lower hemicontinuous (lhc) at \( x \) if whenever \( x \) is in the lower inverse of an open set so is a neighborhood of \( x \). That is, \( \varphi \) is lhc at \( x \) if \( \forall \) open set \( E \) with \( x \in \varphi^L(E) \) \( \exists \) open set \( A \) such that \( x \in A \) and \( A \subseteq \varphi^L(E) \).

The correspondence \( \varphi \) is upper hemicontinuous (respectively, lower hemicontinuous) if it is upper hemicontinuous (respectively, lower hemicontinuous) at every \( x \in X \). Thus, \( \varphi \) is upper hemicontinuous (respectively, lower hemicontinuous) if the upper (respectively, lower hemicontinuous) inverses of open sets are open.

A correspondence is continuous if it is both upper and lower hemicontinuous.

When \( \varphi \) is compact-valued the above definitions of hemicontinuity are equivalent to the following:

Definition 56 A compact-valued correspondence \( \varphi \) is uhc at \( x \) if \( \forall \{x_n\}_{n=1}^{\infty} \) with \( \lim_{n \to \infty} x_n = x \) and \( \forall \{y_n\}_{n=1}^{\infty} \) with \( y_n \in \varphi(x_n) \) for all \( n \) \( \exists \) a convergent subsequence \( \{y_{k_n}\}_{n=1}^{\infty} \) of \( \{y_n\}_{n=1}^{\infty} \) such that \( \lim_{n \to \infty} y_{k_n} \in \varphi(x) \).
A compact-valued correspondence $\varphi$ is lhc at $x$ if $\forall y \in \varphi(x)$ and $\forall \{x_n\}_{n=1}^{\infty}$ with $\lim_{n \to \infty} x_n = x \exists N \geq 1$ and a sequence $\{y_n\}_{n=N}^{\infty}$ such that $\lim_{n \to \infty} y_n = y$ and $y_n \in \varphi(x_n)$ for all $n \geq N$.

If correspondence $\varphi$ is singleton-valued, the upper and the lower inverses of a set coincide and agree with the inverse regarded as a function. Either form of hemicontinuity is equivalent to the continuity of a function. The term “semicontinuity” has been used to mean hemicontinuity, but this usage can lead to confusion when discussing real-valued singleton correspondences. A semicontinuous real-valued function is not a hemicontinuous correspondence unless it is also continuous.

**Definition 57** A correspondence $\varphi$ is closed at $x$ if $\forall \{x_n\}_{n=1}^{\infty}$ with $\lim_{n \to \infty} x_n = x$ and $\forall \{y_n\}_{n=1}^{\infty}$ with $y_n \in \varphi(x_n)$ for all $n$, then $y \in \varphi(x)$. A correspondence is closed if it is closed at every point of its domain, that is, if its graph is closed.

In general, a correspondence may be closed without being upper hemicontinuous, and vice versa. On the other hand, we have:

**Theorem 58** Let $X \subseteq \mathbb{R}^m$ and $Y \subseteq \mathbb{R}^k$. Then
1. If $\varphi$ is upper hemicontinuous and closed-valued, then $\varphi$ is closed.
2. If $Y$ is compact and $\varphi$ is closed, then $\varphi$ is upper hemicontinuous.
3. If $\varphi$ is singleton-valued at $x$ and upper hemicontinuous at $x$, then $\varphi$ is continuous at $x$.

10 **References:**


