Introducción a la Análisis Convexo
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1. Bases

\( \mathbb{R} \) denota los números reales

\( \mathbb{R}^n \) denota el espacio de vectores reales de \( n \) dimensiones (también llamado puntos) \( \mathbf{x} = (x_1, \ldots, x_n) \)

**Definición:** El producto interior de dos vectores \( \mathbf{x} \) y \( \mathbf{y} \) en \( \mathbb{R}^n \) se expresa por

\[
\mathbf{x}' \mathbf{y} = x_1y_1 + \ldots + x_ny_n.
\]

(el producto interior es frecuentemente denotado por \( \langle \mathbf{x}, \mathbf{y} \rangle \)).

**Definición:** El segmento de línea cerrado entre vectores \( \mathbf{x} \) y \( \mathbf{y} \) en \( \mathbb{R}^n \) se define como

\[
\{ \alpha \mathbf{x} + (1-\alpha) \mathbf{y} : \alpha \in [0, 1] \}.
\]

**Definición:** Para cualquier vector no-nulo \( \mathbf{b} \in \mathbb{R}^n \) y cualquier \( \beta \in \mathbb{R} \), los conjuntos

\[
\{ \mathbf{x} : \mathbf{x}'\mathbf{b} \leq \beta \} \quad \text{y} \quad \{ \mathbf{x} : \mathbf{x}'\mathbf{b} \geq \beta \}
\]

son llamados espacios cúbicos cerrados.
The sets 
\[ \{ x : x'b < \beta \} \]  and  \[ \{ x : x'b > \beta \} \]
are called **open half-spaces**.

The set
\[ \{ x : x'b = \beta \} \]
is called a **hyperplane**. The vector \( b \) is called the normal to hyperplane \( H \).

The two closed half-spaces associated with \( H \) are denoted by
\[ H^+ = \{ x : x'b \geq \beta \} \] and \[ H^- = \{ x : x'b \leq \beta \} \].

Note that \( H^+ \cup H^- = \mathbb{R}^n \) and \( H^+ \cap H^- = H \).

**Definition:** A vector sum 
\[ \alpha_1 x^1 + \ldots + \alpha_m x^m \]
is called a **convex combination of** \( x^1, \ldots, x^m \) if 
\( \alpha_1, \ldots, \alpha_m \geq 0 \) and \( \alpha_1 + \ldots + \alpha_m = 1 \).

**Definition:** A subset \( C \) of \( \mathbb{R}^n \) is called **convex** if
\[ \alpha x + (1- \alpha) y \in C \ \forall x, y \in C \ \forall \alpha \in [0, 1]. \]

(that is, for any points \( x, y \) in \( C \), \( C \) contains the line segment between \( x \) and \( y \))
2. The algebra of convex sets

**Theorem 1:** The intersection of an arbitrary collection of convex sets is convex.

*Definition:* A scalar multiple $\alpha C$ of set $C$ in $\mathbb{R}^n$ is defined as

$$\alpha C = \{ \alpha x : x \in C \}.$$

**Theorem 2:** A scalar multiple of a convex set is convex.

*Definition:* A sum of sets $C_1$ and $C_2$ is defined as

$$C_1 + C_2 = \{ x_1 + x_2 : x_1 \in C_1, x_2 \in C_2 \}.$$

**Theorem 3:** If $C_1$ and $C_2$ are convex sets in $\mathbb{R}^n$ then so is their sum $C_1 + C_2$. 
Exercise Set 1

*Exercise 1:* Prove Theorem 1.

*Exercise 2:* Prove Theorem 2.

*Exercise 3:* Prove Theorem 3.

*Exercise 4:* Prove that if $C$ is a convex set and $\alpha_1, \alpha_2 \geq 0$ then $(\alpha_1 + \alpha_2)C = \alpha_1 C + \alpha_2 C$. 
3. Convex functions and their properties

**Definition:** Let $C$ be a convex subset of $\mathbb{R}^n$. A function $f : C \rightarrow \mathbb{R}$ is called **convex** if
\[
f(\alpha x + (1 - \alpha)y) \leq \alpha f(x) + (1 - \alpha)f(y), \quad \forall x, y \in C.
\]

**Definition:** Let $C$ be a convex subset of $\mathbb{R}^n$. A function $f : C \rightarrow \mathbb{R}$ is called **concave** if
\[
f(\alpha x + (1 - \alpha)y) \geq \alpha f(x) + (1 - \alpha)f(y), \quad \forall x, y \in C.
\]

**Definition:** Let $C$ be a convex subset of $\mathbb{R}^n$. A function $f : C \rightarrow \mathbb{R}$ is called **strictly convex** if
\[
f(\alpha x + (1 - \alpha)y) < \alpha f(x) + (1 - \alpha)f(y) \quad \forall x, y \in C \text{ with } x \neq y.
\]

**Theorem 4:** $f$ is a convex function if and only if $-f$ is a concave function.

**Theorem 5:** If $f$ is a convex function, then all of its level sets $\{x \in C : f(x) \leq \beta\}$ and $\{x \in C : f(x) < \beta\}$, where $\beta$ is a scalar, are convex.
**Theorem 6:** (Jensen’s inequality) Let $f$ be a function from convex set $C \subseteq \mathbb{R}^n$ to $\mathbb{R}$.

$f$ is convex on $C$ if and only if

$$f(\alpha_1x^1 + \ldots + \alpha_mx^m) \leq \alpha_1f(x^1) + \ldots + \alpha_mf(x^m)$$

for all $\alpha_1,..,\alpha_m$ such that $\alpha_1,..,\alpha_m \geq 0$ and $\alpha_1+\ldots+\alpha_m=1$.

**Theorem 7:** Let $f$ be a twice continuously differentiable real-valued function on an open interval $(a, b)$. Then $f$ is convex if and only if its second derivative $f''(x)$ is non-negative for all $x \in (a,b)$.

**Definition:** Let $Q$ be an $n \times n$ matrix. $Q$ is **positive semi-definite** if

$$x'(Qx) \geq 0$$

for every $x \in \mathbb{R}^n$.

$Q$ is **positive definite** if

$$x'(Qx) > 0$$

for every $x \in \mathbb{R}^n$ such that $x \neq 0$.

$Q$ is **negative semi-definite** if

$$x'(Qx) \leq 0$$

for every $x \in \mathbb{R}^n$.

$Q$ is **negative definite** if

$$x'(Qx) < 0$$

for every $x \in \mathbb{R}^n$ such that $x \neq 0$. 
These definitions are very frequently to verify. The following result is extremely useful to check for positive and negative (semi-)definiteness. To introduce the result we will need the following definitions.

**Definition:** The *ith principal minor* of $Q$ is the matrix $Q_i$ formed by the first $i$ rows and columns of $Q$.

So, the first principal minor of $Q$ is the matrix $Q_1 = (q_{11})$, the second principal minor is the matrix $Q_2 = \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix}$, and so on.
**Theorem 8:** Consider $n \times n$ matrix $Q$.

1. $Q$ is positive definite if and only if all of $Q$’s principal minors $Q_1, \ldots, Q_n$ have strictly positive determinants.

2. $Q$ is positive semi-definite if and only if all of $Q$’s principal minors $Q_1, \ldots, Q_n$ have nonnegative determinants.

3. $Q$ is negative definite if and only if the determinants of $Q$’s principal minors $Q_1, \ldots, Q_n$ are nonzero and alternate in sign starting with $\det(Q_1) < 0$.

4. $Q$ is negative semi-definite if and only if the determinants of $Q$’s principal minors $Q_1, \ldots, Q_n$ alternate in sign starting with $\det(Q_1) \leq 0$.

**Example:**

$$Q = \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix}.$$
**Definition:** Let \( f(x_1, \ldots, x_n) \) be a twice continuously differentiable real-valued function defined on a set \( C \subseteq \mathbb{R}^n \). **Hessian matrix** \( H \) of function \( f \) is defined as

\[
H(f) = \begin{bmatrix}
\frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\
\frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2}
\end{bmatrix}
\]

Note that the Hessian matrix is a symmetric \( n \times n \) matrix.

**Theorem 9:** Let \( f \) be a twice continuously differentiable real-valued function on an open convex set \( C \subseteq \mathbb{R}^n \). Then

1. \( f \) is convex on \( C \) if and only if its Hessian matrix is positive semi-definite for every \( x \in C \).
2. \( f \) is concave on \( C \) if and only if its Hessian matrix is negative semi-definite for every \( x \in C \).
3. \( f \) is strictly convex on \( C \) if and only if its Hessian matrix is positive definite for every \( x \in C \).
4. \( f \) is strictly concave on \( C \) if and only if its Hessian matrix is negative definite for every \( x \in C \).
Definition: A function $f$ on $\mathbb{R}^n$ is said to be **positively homogeneous of degree 1** if for every $x \in \mathbb{R}^n$ one has

$$f(\alpha x) = \alpha f(x) \text{ for all } \alpha \in (0, \infty).$$

Theorem 10: A positively homogeneous function $f$ on $\mathbb{R}^n$ is convex if and only if

$$f(x + y) \leq f(x) + f(y), \quad \forall x, y \in \mathbb{R}^n.$$

Theorem 11: A real-valued convex function $f$ on $\mathbb{R}^n$ is continuous.

Theorem 12: A real-valued convex function $f$ on $\mathbb{R}^n$ is differentiable everywhere except for (at most) a set of measure zero.
Exercise Set 2

Exercise 5: Prove Theorem 4.

Exercise 6: Prove Theorem 5.

Exercise 7: Prove Theorem 6.

Exercise 8: Prove Theorem 7.

Exercise 9: Check whether the following matrices are positive definite, positive semi-definite, negative definite, negative semi-definite or none of the above:

(i) \( Q = \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix} \); (ii) \( Q = \begin{pmatrix} -2 & 4 \\ 4 & -8 \end{pmatrix} \);

(iii) \( Q = \begin{pmatrix} -2 & 2 \\ 2 & -4 \end{pmatrix} \); (iv) \( Q = \begin{pmatrix} 2 & 4 \\ 4 & 3 \end{pmatrix} \).

Exercise 10: Check whether the following function is convex, strictly convex, concave, strictly concave, or none of the above:

\[ f(x_1, x_2) = x_1^{0.25}x_2^{0.5}. \]

Exercise 11: Prove Theorem 10.
4. Functional operations on convex functions

**Theorem 13:** Let \( f \) be a convex function from \( \mathbb{R}^n \) to \( \mathbb{R} \) and let \( \varphi \) be a non-decreasing convex function from \( \mathbb{R} \) to \( \mathbb{R} \). Then, \( h(x) = \varphi(f(x)) \) is convex on \( \mathbb{R}^n \).

**Corollary of Theorem 13:** If \( f \) is a real-valued convex function on \( \mathbb{R}^n \) and \( a \) is a non-negative constant, then \( af \) is convex on \( \mathbb{R}^n \).

**Theorem 14:** If \( f_1 \) and \( f_2 \) are real-valued convex functions on \( \mathbb{R}^n \), then \( f_1 + f_2 \) is convex on \( \mathbb{R}^n \).

**Theorem 15:** The pointwise supremum of an arbitrary collection of convex functions is convex. That, if \( f_i \)‘s (for \( i \in I \)) are convex on \( \mathbb{R}^n \) then function
\[
 f(x) = \sup \{ f_i(x) : i \in I \}
\]
is convex on \( \mathbb{R}^n \).
**Theorem 16:** If $f(x; y)$ is convex in $x$ for each $y \in A$, then

$$g(x) = \sup \{ f(x; y) : y \in A \}$$

is a convex function.

**Theorem 17:** If $f(x; y)$ is convex in $(x; y)$ and $C$ is a convex set, then

$$g(x) = \inf \{ f(x; y) : y \in C \}$$

is a convex function.

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**Exercise Set 3**

**Exercise 12:** Prove Theorem 13.

**Exercise 13:** Prove Theorem 14.

**Exercise 14:** Prove Theorem 15.

**Exercise 15:** Prove Theorem 16.

**Exercise 16:** Prove Theorem 17.
5. Quasiconvex functions.

**Definition:** Let $C$ be a convex subset of $\mathbb{R}^n$. A function $f : C \to \mathbb{R}$ is called **quasiconvex** if its lower level sets $\{x \in C : f(x) \leq \beta\}$, where $\beta$ is an arbitrary scalar, are convex.

**Definition:** Let $C$ be a convex subset of $\mathbb{R}^n$. A function $f : C \to \mathbb{R}$ is called **quasiconcave** if its upper level sets $\{x \in C : f(x) \geq \beta\}$, where $\beta$ is an arbitrary scalar, are convex.

**Theorem 18:** Function $f$ is quasiconvex if and only if function $-f$ is quasiconcave.

**Theorem 19:** Function $f$ is quasiconvex iff
\[ f(\alpha x + (1 - \alpha)y) \leq \min \{f(x), f(y)\} \quad \forall x, y \in C \quad \forall \alpha \in [0, 1]. \]
Exercise Set 4

Exercise 17: Prove Theorem 18.

Exercise 18: Prove Theorem 19.

Exercise 19: State and prove an analogue of Theorem 19 for quasiconcave functions.

Exercise 20: Demonstrate that $f(x) = \sqrt{|x|}$ is quasiconvex.

Exercise 21: Demonstrate that $f(x_1, x_2) = x_1 x_2$ is quasiconcave.

Exercise 22: Consider a continuous increasing function $f: \mathbb{R} \rightarrow \mathbb{R}$. Decide whether $f$ is concave, convex, quasiconcave, quasiconvex, or none of the above. Justify your answer.

*Definition:* Let $C_1$ and $C_2$ be nonempty sets in $\mathbb{R}^n$. A hyperplane $H$ is said to separate $C_1$ and $C_2$ if $C_1$ is contained in one of the closed half-spaces associated with $H$ while $C_2$ is contained in the opposite closed half-space.

*Definition:* Let $C_1$ and $C_2$ be nonempty sets in $\mathbb{R}^n$. A hyperplane $H$ is said to properly separate $C_1$ and $C_2$ if $H$ separates $C_1$ and $C_2$ and $C_1$ and $C_2$ are not both contained in $H$.

*Definition:* Let $C_1$ and $C_2$ be nonempty sets in $\mathbb{R}^n$. A hyperplane $H$ is said to strongly separate $C_1$ and $C_2$ if $C_1 + \varepsilon B$ is contained in one of the open half-spaces associated with $H$ while $C_2 + \varepsilon B$ is contained in the opposite open half-space, where $B \equiv \{x : |x| \leq 1\}$.

Recall that $H = \{x : x'b = \beta\}$, $H^+ = \{x : x'b \geq \beta\}$ and $H = \{x : x'b \leq \beta\}$. 
**Theorem 20:** Let $C_1$ and $C_2$ be nonempty sets in $R^n$. There exists a hyperplane properly separating $C_1$ and $C_2$ if and only if there exists a vector $b$ such that

(i) $\inf \{x'b : x \in C_1\} \geq \sup \{x'b : x \in C_2\}$ and

(ii) $\sup \{x'b : x \in C_1\} > \inf \{x'b : x \in C_2\}$.

**Theorem 21:** Let $C_1$ and $C_2$ be nonempty disjoint closed convex sets in $R^n$. Then there exists a hyperplane strongly separating $C_1$ and $C_2$.

**Corollary of Theorem 21:** Let $C$ be a closed convex set in $R^n$ and let $x \not\in C$. Then there exists a hyperplane strongly separating $\{x\}$ and $C$. That is, there exist $b \in R^n$ and $\beta \in R$ such that $x'b > \beta$ and $y'b < \beta$ for all $y \in C$.

**Theorem 22:** Let $C_1$ and $C_2$ be nonempty disjoint convex sets in $R^n$. Then there exists a hyperplane separating $C_1$ and $C_2$. 
**Definition:** Let $C$ be a convex set in $R^n$. A **supporting half-space** to $C$ is a closed half-space which contains $C$ and contains a point in the boundary of $C$. A **supporting hyperplane** to $C$ is the boundary of a supporting half-space to $C$. More formally, $H = \{ x : x'b = \beta \}$ is a supporting hyperplane to $C$ if

i) if either $y'b \leq \beta \quad \forall \ y \in C$ or $y'b \geq \beta \quad \forall \ y \in C$; and

ii) $\exists \ y$ in the boundary of $C$ such that $y'b = \beta$.

(Note that the above definition is slightly different from the “common” definition where a supporting half-space is defined as a closed half-space which contains $C$ and has a point of $C$ in its boundary.)

**Theorem 23:** *(Supporting Hyperplane Theorem)* Let $C$ be a convex set in $R^n$. Then there exists a supporting hyperplane at every boundary point of $C$.

**Exercise Set 5**

*Exercise 23:* Prove Theorem 20.

*Exercise 24:* Prove Theorem 21.

*Exercise 25:* Prove Theorem 22.

*Exercise 26:* Prove Theorem 23.
7. Convex hull.

*Definition:* Convex hull of a set $C$, denoted by $\text{conv}(C)$, is the intersection of all convex sets containing $C$.

**Theorem 24:** $\text{conv}(C)$ consists of the set of all convex combinations of points from $C$.

**Theorem 25:** The closure of the convex hull of $C$ ($\text{cl}(\text{conv}(C))$) is equal to the intersection of all the closed half-spaces containing $C$. 
8. Conjugates of convex functions.

*Definition:* Let $C$ be a convex subset of $R^n$. The **extended real line** is defined as

$$[-\infty, +\infty] = \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}.$$  

An **extended real-valued function** $f$ is defined as $f: C \rightarrow [-\infty, +\infty]$.

*Definition:* Let $f$ be an extended real-valued function on $R^n$. The **conjugate** of $f$ is function $f^*(x^*)$ on $R^n$ defined by

$$f^*(x^*) = \sup \{ x^* x^* - f(x) : x \in R^n \}.$$  

**Theorem 26:** The conjugate of $f^*$ is $f$.

**Theorem 27:** The conjugate of a convex function is convex.

**Examples:** profit, revenue, and cost functions
Exercise Set 6


Exercise 28: Prove Theorem 27.

Exercise 29: Determine the conjugate of function
\[ f(x) = \exp(x), \quad x \in \mathbb{R}. \]

*Definition:* Let $C$ be a nonempty set in $\mathbb{R}^n$. The (upper) **support function** of set $C$ is defined as

$$\delta^*(x|C) = \sup \{ x'y : y \in C \}.$$  

(For the lower support function, sup is replaced by inf).

*Definition:* Let $C$ be a nonempty set in $\mathbb{R}^n$. The **indicator function** of set $C$ is defined as

$$\delta(x|C) = \begin{cases} 0 & \text{if } x \in C \\ +\infty & \text{if } x \notin C \end{cases}.$$  

*Theorem 27:* The indicator function and the support function of a closed convex set are conjugate to each other.

*Examples:* profit, revenue, and cost functions