

Theory of Interest - Formula Sheet II

1. **Continuous annuities.** If the payments are being made continuously at the rate $f(t)$ at exact moment t , then the present value of an n -period continuous varying annuity is

$$\int_0^n f(t)e^{-\int_0^t \delta_r dr} dt,$$

where δ_r is the force of interest. Under compound interest this becomes

$$\int_0^n f(t)v^t dt.$$

Special cases:

- If $f(t) = 1$ then the present value is

$$\bar{a}_{\overline{n}|} = \int_0^n v^t dt = \frac{1 - v^n}{\delta}.$$

The accumulated value is

$$\bar{s}_{\overline{n}|} = (1 + i)^n \bar{a}_{\overline{n}|} = \int_0^n (1 + i)^{n-t} dt = \frac{(1 + i)^n - 1}{\delta}.$$

- If $f(t) = t$ (continuously increasing annuity) then the present value is

$$(\bar{Ia})_{\overline{n}|} = \int_0^n tv^t dt = \frac{\bar{a}_{\overline{n}|} - nv^n}{\delta}.$$

2. **Varying annuity - arithmetic progression.** Annuity immediate with a term of n periods in which payments begin at P and increase by Q per period thereafter:

$$\text{Present Value} = Pa_{\overline{n}|} + Q \frac{a_{\overline{n}|} - nv^n}{i}.$$

$$\text{Accumulated Value} = Ps_{\overline{n}|} + Q \frac{s_{\overline{n}|} - n}{i}.$$

Special case $P = Q = 1$:

$$\text{Present Value} = (Ia)_{\overline{n}|} = \frac{\ddot{a}_{\overline{n}|} - nv^n}{i}.$$

$$\text{Accumulated Value} = (Is)_{\overline{n}|} = (Ia)_{\overline{n}|} (1 + i)^n = \frac{\ddot{s}_{\overline{n}|} - n}{i}.$$

Special case $P = n, Q = -1$:

$$\text{Present Value} = (Da)_{\overline{n}|} = \frac{n - a_{\overline{n}|}}{i}.$$

$$\text{Accumulated Value} = (Ds)_{\overline{n}|} = (Da)_{\overline{n}|} (1 + i)^n = \frac{n(1 + i)^n - s_{\overline{n}|}}{i}.$$

3. **Varying annuity - geometric progression.** The present value of an annuity immediate with a term of n periods in which the first payment is 1 and successive payments increase in geometric progression with a common ratio $1 + k$ is

$$\frac{1 - \left(\frac{1+k}{1+i}\right)^n}{i - k}, \quad i \neq k.$$

4. **Dollar-weighted rate of interest.** The rate i is the solution of an exact equation of value:

$$I = iA + \sum_t C_t \cdot {}_{1-t}i_t,$$

where

$I = B - A - C$ - the interest.

A - The amount in the fund at the beginning of the period.

B - The amount in the fund at the end of the period.

C_t - The net amount of principal contributed at time t .

C - The total amount of principal contributed during the period.

${}_{1-t}i_t$ - the rate of interest for the period $(t, 1)$. Under compound interest use ${}_{1-t}i_t = (1 + i)^{1-t} - 1$. For an approximation (simple interest) use ${}_{1-t}i_t = (1 - t)i$.

5. **Time weighted rate of interest.** Here, the rate i is equal to

$$i = (1 + j_1)(1 + j_2) \cdots (1 + j_m) - 1,$$

where

$$j_k = \frac{B'_k}{B'_{k-1} + C'_{k-1}}, \quad k = 1, 2, \dots, m,$$

is the rate over the subinterval (t_{k-1}, t_k) and

C'_k - The net amount of principal contributed at time t_k .

B'_k - The fund value (immediately before each contribution) at time t_k .

6. **Loans with varying series of payments.** Basic equation of value:

$$L = \sum_t v^t R_t,$$

where

L - The loan amount.

R_t - installment payment (including principal and interest) at time t .

For the sinking fund method we have:

$$L = \sum_t R_t(1 + j)^{n-t} - iLs_{\overline{n}|j},$$

where L and R_t are as above and

i - rate of interest paid on the loan.

j - rate of interest earned on the sinking fund.