OPERATOR ALGEBRAS

Some historical markers

-1930: John von Neumann (the same Hungarian guy from computer science, game theory and economics) defines rings of operators, now called von Neumann algebras.

-1943: Gelfand and Neumark show that Banach algebras with an adjoint $*$ satisfying certain properties (now called C*-algebras) are $*$ isomorphic to norm-closed self-adjoint subalgebras of $\mathcal{B}(H)$.

-1947: Segal relates harmonic analysis on locally compact groups to (non-commutative) self-adjoint operator algebras acting on a Hilbert space. Also, Segal treats the foundations of quantum mechanics from the point of view of operator algebras.

-1950's: The elaboration of the viewpoint that C*-algebras are non-commutative versions of $C_0(X)$ and von Neumann algebras are non-commutative versions of $L^\infty(X, \mu)$.

-1967: Haag, Hugenholtz and Winnink relate quantum statistical mechanics and operator algebras by introducing the KMS condition.

-1970's: Several major ideas from algebraic topology (like K-theory and extensions) were incorporated.

-1980's: Connes starts his vast non-commutative geometry program. Jones studies subfactors and discovers a new knot invariant. Voiculescu initiates the theory of free probability and free entropy.

-1990's and 2000's: Big progress in the classification theory of C*-algebras in terms of K-theoretical invariants.

Definitions and Examples

A C*-algebra is a complex Banach algebra A with involution $*$ such that $||a^*a|| = ||a||^2$. It is a non-commutative ring of operators (think infinite matrices), in fact $A \subset \mathcal{B}(H)$ for H a Hilbert space (think infinite dimensional Euclidean space).

1) $C(X) = \{f : X \to \mathbb{C} \mid f$ continuous$\}$ with norm $||f|| = \sup|f(x)|$ and involution $f^*(x) = \overline{f(x)}$, where X is a compact Hausdorff space. Any commutative unital C*-algebra is isomorphic to some $C(X)$.

2) M_n the set of $n \times n$ matrices M with complex entries, $||M|| = \sup\{|Mv| : |v| = 1\}$, where $|v|$ is the vector length, $M^* = \overline{M^t}$. Every finite dimensional C*-algebra is a direct sum of matrices.

3) $\mathcal{B}(H)$, the bounded linear operators T on a Hilbert space with involution given by $\langle Tx, y\rangle = \langle x, T^*y\rangle$.

4) Inductive limits of finite dimensional C*-algebras, like the compact operators (limits of finite rank operators)

$$\mathbb{C} \to M_2 \to M_3 \to ...$$

or the CAR algebra or Fermion algebra (used to represent the canonical anti-commutation relations for systems of particles obeying Fermi statistics)

$$M_2 \to M_4 \to M_8 \to ..., a \mapsto \begin{bmatrix} a & \\ & a \end{bmatrix}.$$

5) $C(X, M_n) = \{f : X \to M_n \mid \ f \text{ continuous}\}$ and inductive limits like the Bunce-Deddens algebras

$$C(\mathbb{T}) \to C(\mathbb{T}, M_2) \to C(\mathbb{T}, M_4) \to ..., z \mapsto \begin{bmatrix} 0 & 1 \\ z & 0 \end{bmatrix}.$$

6) The convolution algebra $C^*(G)$ of a locally compact group (or groupoid) G. For $a, b \in C_c(G)$, define

$$(a \cdot b)(g) = \int a(h) b(h^{-1} g) dh,$$

$$a^*(g) = \overline{a(g^{-1})}, ||a|| = \int |a(g)| dg,$$

and then take a completion.

For G abelian, $C^*(G) \cong C_0(\widehat{G})$ by Fourier transform. For example, $C^*(\mathbb{Z}) \cong C(\mathbb{T})$.

For $G = \mathbb{Z} \rtimes \mathbb{Z}_2$ the infinite dihedral group, $C^*(G)$ is isomorphic to

$$C(\mathbb{T}) \rtimes \mathbb{Z}_2 \cong \{f : [0,1] \to M_2 \mid \ f(0), f(1) \text{ diagonal}\}.$$

7) Graph C*-algebras. Let (E^0, E^1, r, s) be an oriented graph. Its C*-algebra is generated by mutually orthogonal projections P_v for $v \in E^0$ and partial isometries T_e for $e \in E^1$ such that

$$T_e^* T_e = P_{s(e)}, \;\; P_v = \sum_{r(e)=v} T_e T_e^*.$$

Application

In quantum field theory, one describes a physical system by a unital C*-algebra A. The self-adjoint elements of A ($x \in A$ with $x^* = x$) are thought of as the observables, or the measurable quantities, of the system. A state of the system is a positive functional on A (a $\mathbb{C}$-linear map $\phi : A \to \mathbb{C}$ with $\phi(aa^*) > 0$ for all $a \in A$) such that $\phi(1_A) = 1_A$. If the system is in state ϕ, then $\phi(x)$ is the expected value of the observable x.

References

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W. Arveson, An Invitation to C*-Algebra, Springer 1976. A good introduction for those with a knowledge of basic functional analysis.

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