Group actions on topological graphs

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(work in progress)

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Outline

- We define the action of a locally compact group G on a topological graph E.
- This action induces a natural action of G on the C^*-correspondence $\mathcal{H}(E)$ and on the graph C^*-algebra $C^*(E)$.
- If the action is free and proper, then E/G is a topological graph, and $C^*(E) \rtimes_r G$ is strongly Morita equivalent to $C^*(E/G)$.
- We define the skew product $E \times_c G$ of a topological graph E by a locally compact group G via a cocycle $c : E^1 \to G$.
- If G is abelian, there is a dual action on $C^*(E)$, and $C^*(E) \rtimes \hat{G} \cong C^*(E \times_c G)$.
- We also define the fundamental group and the universal covering of a topological graph.

Topological graphs and their C^*-algebras

- Let $E = (E^0, E^1, s, r)$ be a topological graph. Recall that E^0 (vertices) and E^1 (edges) are locally compact spaces, $s, r : E^1 \to E^0$ are continuous maps, and s is a local homeomorphism.
- The C*-algebra $C^*(E)$ is the Cuntz-Pimsner algebra $\mathcal{O}_{\mathcal{H}}$ of the C^*-correspondence $\mathcal{H} = \mathcal{H}(E)$ over $A = C_0(E^0)$, obtained as a completion of $C_c(E^1)$ using

$$\langle \xi, \eta \rangle(v) = \sum_{s(e)=v} \overline{\xi(e)}\eta(e),\ \xi, \eta \in C_c(E^1)$$

$$(\xi \cdot f)(e) = \xi(e)f(s(e)),\ (f \cdot \xi)(e) = f(r(e))\xi(e).$$

Examples

- Let $E^0 = E^1 = \mathbb{T}, s(z) = z$, and $r(z) = e^{2\pi i\theta}z$ for $\theta \in [0, 1]$ irrational. Then $C^*(E) \cong A_\theta$, the irrational rotation algebra.
- Let $E^0 = E^1 = X$, for X a locally compact metric space, let $s = id$ and let $r = h : X \to X$ be a homeomorphism.
  Then $C^*(E) \cong C_0(X) \rtimes \mathbb{Z}$, since $C^*(E)$ is the universal C^*-algebra generated by $C_0(X)$ and a unitary u satisfying $\hat{h}(f) = u^*fu$ for $f \in C_0(X)$, where $\hat{h}(f) = f \circ h$.
- Let $n \in \mathbb{N} \setminus \{0\}$ and $m \in \mathbb{Z} \setminus \{0\}$. Take

$$E^0 = E^1 = \mathbb{T}, s(z) = z^n, r(z) = z^m.$$

  If $m \notin n\mathbb{Z}$, then $C^*(E)$ is simple and purely infinite.

The Cayley graph

- The Cayley graph of a finitely generated locally compact group.
  A locally compact group G is *finitely generated* if there is a finite subset $S \subset G$ such that $G = \overline{\langle S \rangle}$. If $S = \{h_1, h_2, ..., h_n\}$, define the Cayley graph $E = E(G, S)$ with $E^0 = G, E^1 = S \times G, s(h, g) = g$, and $r(h, g) = gh$.
- Then $E(G, S)$ becomes a topological graph. For G discrete finitely generated, we get the usual notion of Cayley graph. (The Cayley graph may change if we change the set of generators).
- For $G = (\mathbb{R}, +)$ and $S = \{1, \theta\}$, where $\theta < 0$ is irrational, the Cayley graph E has $E^0 = \mathbb{R}$, $E^1 = \{1, \theta\} \times \mathbb{R}$,

  $$s(1, x) = x, r(1, x) = x + 1, s(\theta, x) = x, r(\theta, x) = x + \theta.$$

  Its C*-algebra is simple and purely infinite, isomorphic to $\mathcal{O}_2 \rtimes_\alpha \mathbb{R}$. Here $\alpha_t(V_0) = e^{it\theta} V_0,\ \alpha_t(V_1) = e^{it} V_1$ for $t \in \mathbb{R}$ and V_0, V_1 are the standard generators of the Cuntz algebra $\mathcal{O}_2$.

Skew products

- Skew products of topological graphs. Let $E = (E^0, E^1, s, r)$ be a graph, let G be a locally compact group, and let $c : E^1 \to G$ be continuous. Define the *skew product graph* $E \times_c G = (E^0 \times G, E^1 \times G, \tilde{s}, \tilde{r})$, where

$$\tilde{s}(e, g) = (s(e), g), \quad \tilde{r}(e, g) = (r(e), c(e)g).$$

- Then $E \times_c G$ becomes a topological graph using the product topology. If E has one vertex and n loops $\{e_1, ..., e_n\}$ and if G has a set of generators $S = \{h_1, ..., h_n\}$ such that $c(e_i) = h_i, i = 1, ..., n$ then we get the Cayley graph $E(G, S)$.

Graph morphisms

- Let E, F be two topological graphs. A graph morphism $\phi : E \to F$ is a pair of continuous maps $\phi = (\phi^0, \phi^1)$ such that the diagram

$$\begin{array}{ccccc} E^0 & \xleftarrow{s} & E^1 & \xrightarrow{r} & E^0 \\ {\scriptstyle \phi^0}\downarrow & & {\scriptstyle \phi^1}\downarrow & & {\scriptstyle \phi^0}\downarrow \\ F^0 & \xleftarrow{s} & F^1 & \xrightarrow{r} & F^0 \end{array}$$

  is commutative.
- A graph morphism ϕ is a *graph covering* if both ϕ^0, ϕ^1 are covering maps.
- An isomorphism is a graph morphism $\phi = (\phi^0, \phi^1)$ such that ϕ^i are homeomorphisms for $i = 0, 1$. It follows that $\phi^{-1} = ((\phi^0)^{-1}, (\phi^1)^{-1})$ is also a graph morphism.

Group actions

- A locally compact group G acts on E if there are continuous maps $\lambda^i : G \times E^i \to E^i$ for $i = 0, 1$ such that $g \mapsto \lambda_g$ is a homomorphism from G into $Aut(E)$.
- The action λ is called free if $\lambda_g^0(v) = v$ for some $v \in E^0$ implies $g = 1_G$. In this case the action of G is also free on E^1.
- The action is called *proper* if the maps $G \times E^0 \to E^0 \times E^0, (g, v) \mapsto (\lambda_g^0(v), v)$ and $G \times E^1 \to E^1 \times E^1, (g, e) \mapsto (\lambda_g^1(e), e)$ are proper. (It is sufficient to require properness of the first map).
- A group G acts freely and properly on a skew product $E \times_c G$ by $\lambda_g^0(v, h) = (v, gh)$ and $\lambda_g^1(e, h) = (e, gh)$.

The quotient graph

- For a free and proper action of G on E, the quadruple $E/G = (E^0/G, E^1/G, s, r)$ becomes a topological graph, where E^i/G are the orbit spaces for $i = 0, 1$ and $s(\hat{e}) = \widehat{s(e)}, r(\hat{e}) = \widehat{r(e)}$.
- Here $q : E \to E/G, q(x) = \hat{x}$ is the quotient morphism, called a principal G-bundle map. For G discrete, this is a graph covering.
- A finitely generated locally compact group G with generating set S acts freely and properly on its Cayley graph $E = E(G, S)$ by $\lambda^0_g(g') = gg'$ and $\lambda^1_g(h, g') = (h, gg')$. The quotient graph E/G has $|S|$ loops and one vertex.
- For a skew product $E \times_c G$, the map $\phi : E \times_c G \to E,\ \phi(g, x) = x$ is a principal G-bundle map, and the quotient graph is isomorphic to E.
- Unlike the discrete case, not all principal G-bundles come from skew products.

The fundamental group

- The *geometric realization* of a topological graph E is

$$R(E) := E^1 \times [0,1] \sqcup E^0 / \sim,$$

where $(e,0) \sim s(e)$ and $(e,1) \sim r(e)$ (a kind of double mapping torus).

- If the group G acts on the topological graph E, then G acts on $R(E)$ by

$$g \cdot (e,t) = (\lambda_g^1(e), t), e \in E^1, t \in [0,1], \quad g \cdot v = \lambda_g^0(v), v \in E^0.$$

- The fundamental group $\pi_1(E)$ is by definition $\pi_1(R(E))$. The universal covering $\tilde{E}$ of E is a simply connected graph which covers E.
- The group $\pi_1(E)$ acts freely on $\tilde{E}$, and the orbit space $\tilde{E}/G$ is isomorphic to E. Any subgroup H of $\pi_1(E)$ will determine an intermediate covering of E, by taking the graph $\tilde{E}/H$.

Coverings

Proposition.

Let E be such that $R = R(E)$ has a universal covering space $\tilde{R}$. Then $\tilde{R}$ is homeomorphic to the geometric realization of a simply connected topological graph $\tilde{E}$, which covers E.

Proof.

Let $\pi : \tilde{R} \to R$ be the canonical map, let $\tilde{E}^0 = \pi^{-1}(E^0)$, and let $\tilde{E}^1 = \pi^{-1}(E^1 \times \{1/2\})$. In order to define the range and the source maps, we use the unique path lifting property of the map π. □

Corollary

To obtain other coverings of a graph E as above, we take a subgroup H of $\pi_1(E)$, and take the corresponding topological graph of the quotient space $\tilde{R}/H$.

Examples

- Let E with $E^0 = E^1 = \mathbb{T}$ and with $s(z) = z, r(z) = e^{2\pi i\theta}z$ for θ irrational. The geometric realization is homeomorphic to $\mathbb{T}^2$, hence $\pi_1(E) \cong \mathbb{Z}^2$.
- Consider the graph F with $F^0 = F^1 = \mathbb{R} \times \mathbb{Z}, s(t,k) = (t,k), r(t,k) = (t+\theta, k+1)$. Then $\mathbb{Z}^2$ acts on F by $(m_1, m_2) \cdot (t,k) = (t + m_1, k + m_2)$, and $F/\mathbb{Z}^2 \cong E$. The graph F is the universal covering graph of E.
- Let $h : X \to X$ be a homeomorphism, and let E with $E^0 = E^1 = X$, $s = id$ and $r = h$. The geometric realization of E is homeomorphic to the mapping torus of h.
- The graph $\tilde{E}$ has $\tilde{E}^0 = \tilde{E}^1 = \tilde{X} \times \mathbb{Z}$, where $\tilde{X}$ is the universal covering space of X. The source and range maps are $s(y,k) = (y,k)$, $r(y,k) = (\tilde{h}(y), k+1)$, where $\tilde{h} : \tilde{X} \to \tilde{X}$ is a lifting of h.
- Then $\pi_1(E) \cong \pi_1(X) \rtimes \mathbb{Z}$. The action of $\pi_1(X) \rtimes \mathbb{Z}$ on $\tilde{X} \times \mathbb{Z}$ is given by $(g, m) \cdot (y, k) = (g \cdot \tilde{h}^m(y), k + m)$.

Examples cont'd

- Let again $E^0 = E^1 = \mathbb{T}$ with $s(z) = z^p, r(z) = z^q$ for p, q positive integers. Then $R(E)$ is obtained from a cylinder, where the two boundary circles are identified using the maps s and r. Then $\pi_1(E)$ is isomorphic to the Baumslag-Solitar group

$$B(p, q) = \langle a, b \mid \ ab^p a^{-1} = b^q \rangle.$$

- $B(p, q)$ is a HNN-extension of $\pi_1(\mathbb{T})$. It is the quotient of the free product $\pi_1(\mathbb{T}) * \mathbb{Z}$ by the relation $as_*(b)a^{-1} = r_*(b)$, where a is the generator of $\mathbb{Z}$, $b \in \pi_1(\mathbb{T})$, and $s_*, r_* : \pi_1(\mathbb{T}) \hookrightarrow \pi_1(\mathbb{T})$. For $p = 1$ or $q = 1$, this group is a semi-direct product and it is amenable.
- The universal covering space of $R(E)$ is obtained from the Cayley graph of $B(p, q)$ by filling out the squares. It is the cartesian product $T \times \mathbb{R}$, where T is the Bass-Serre tree of $B(p, q)$.

Group actions on C^*-correspondences

- A group G acts on a $A-A$ C^*-correspondence $\mathcal{H}$ if there is a map $G \times \mathcal{H} \to \mathcal{H}, \quad (g, \xi) \mapsto g \cdot \xi$ such that $g \mapsto g \cdot \xi$ is continuous, $\xi \mapsto g \cdot \xi$ is linear, and if G acts on A by $*$-automorphisms such that $\langle g{\cdot}\xi, g{\cdot}\eta\rangle = g{\cdot}\langle \xi, \eta\rangle,\ g{\cdot}(\xi a) = (g{\cdot}\xi)(g{\cdot}a),\ g{\cdot}(\varphi(a)\xi) = \varphi(g{\cdot}a)(g{\cdot}\xi)$.
- An action of G on the C^*-correspondence $\mathcal{H}$ defines an action on the Cuntz-Pimsner algebra $\mathcal{O}_{\mathcal{H}}$ since all defining relations are equivariant.

Proposition.

If G acts on the topological graph $E = (E^0, E^1, s, r)$, then G acts on the C^*-correspondence $\mathcal{H} = \mathcal{H}(E)$ and hence on $C^*(E)$.

Proof.

Define $g \cdot \xi(e) = \xi(g^{-1}e)$ for $\xi \in C_c(E^1)$, $g \cdot f(v) = f(g^{-1}v)$ for $f \in C_0(E^0)$. Then this action is compatible with the bimodule structure since s and r are equivariant. □

Proper actions

- The action α of a locally compact group G on a C^*-algebra A is *proper* if there is a dense α-invariant $*$-subalgebra A_0 of A such that
- for every $a, b \in A_0$ the functions

$$x \mapsto a\alpha_x(b) \text{ and } x \mapsto \Delta(x)^{-1/2} a\alpha_x(b)$$

  are integrable on G, and
- for all $a, b \in A_0$ there exists $\langle a, b\rangle_r \in M(A_0)$, where

$$M(A_0) := \{m \in M(A) : a \in A_0 \Rightarrow ma \in A_0\}$$

  such that

$$c\langle a, b\rangle_r = \int_G c\alpha_x(a^*b)dx \text{ for all } c \in A_0.$$

Proper action cont'd

- For such an action,

$$A^\alpha := \overline{span}\{\langle a, b\rangle_r : a, b \in A_0\} \subset M(A)$$

  is called the *generalized fixed-point algebra*.
- Define a (left) inner product on A_0 with values in $A \rtimes_{\alpha,r} G$ by

$$_\ell\langle a, b\rangle(x) = \Delta(x)^{-1/2} a\alpha_x(b^*).$$

- The set

$$I := \overline{span}\{_\ell\langle a, b\rangle : a, b \in A_0\}$$

  is an ideal in $A \rtimes_{\alpha,r} G$, and the closure $\mathcal{Z}$ of A_0 in the norm $\|a\|^2 := \|\langle a, a\rangle_r\|$ is an $I - A^\alpha$ imprimitivity bimodule.
- The action is called *saturated* if $I = A \rtimes_{\alpha,r} G$.

The main results

Theorem

If G acts freely and properly on the topological graph E, then G acts properly on $C^*(E)$ and the action is saturated. Moreover, $C^*(E) \rtimes_r G$ and $C^*(E/G)$ are strongly Morita equivalent.

Sketch of proof. (under construction) Since G acts freely and properly on E^0 and there is an equivariant map $C_0(E^0) \to M(C^*(E))$, it follows that the action of G on $C^*(E)$ is proper and saturated with respect to

$$A_0 = C_c(E^0)C_c(E^1)C_c(E^0).$$

To prove that the generalized fixed point algebra is isomorphic to $C^*(E/G)$, we construct maps $\pi : C_0(E^0/G) \to C_b(E^0), \pi(f)(v) = f(\hat{v})$ and $\tau : C_c(E^1/G) \to C_b(E^1), \tau(\xi)(e) = \xi(\hat{e})$ which induce a representation of $C^*(E/G)$ into $M(C^*(E))$.

The main results cont'd

Recall that G acts freely and properly on $E \times_c G$ and that $(E \times_c G)/G = E$. We have

Corollary

The C^*-algebras $C^*(E \times_c G) \rtimes_r G$ and $C^*(E)$ are strongly Morita equivalent. In particular, for a finitely generated locally compact group G with generators $S = \{h_1, h_2, ..., h_n\}$ and Cayley graph $E(G, S)$, we get that $C^*(E(G, S)) \rtimes_r G$ is strongly Morita equivalent to the Cuntz algebra $\mathcal{O}_n$.

Corollary

If G is abelian, $c : E^1 \to G$ induces an action α^c of $\hat{G}$ on $C^*(E)$ such that $(\alpha^c_\chi \xi)(e) = \langle \chi, c(e) \rangle \xi(e)$ for $\xi \in C_c(E^1)$ and $\chi \in \hat{G}$. Then

$$C^*(E) \rtimes_{\alpha^c} \hat{G} \cong C^*(E \times_c G).$$

References

- A. an Huef, I. Raeburn, D. Williams, *A symmetric imprimitivity theorem for commuting proper actions*, Canad. J. Math. 57 (2005) 983–1011.
- C. Anantharaman-Delaroche, J. Renault, Amenable groupoids.
- M. Bridson, A. Haefliger, Metric spaces of non-positive curvature.
- T. Katsura, *A class of C^*-algebras generalizing both graph algebras and homeomorphism C^*-algebras I, Fundamental results*, Trans. Amer. Math. Soc. 356 (2004), no. 11, 4287–4322.
- T. Katsura, *A class of C^*-algebras generalizing both graph algebras and homeomorphism C^*-algebras IV, pure infiniteness*, J. Funct. Anal. 254 (2008), no. 5, 1161–1187.
- T. Katsura, *Continuous graphs and crossed products of Cuntz algebras*, Recent aspects of C^*-algebras (Japanese) (Kyoto, 2002). Sūrikaisekikenkyūsho Kōkyūroku No. 1291 (2002), 73–83.

References cont'd

- A. Kishimoto, A. Kumjian, *Crossed products of Cuntz algebras by quasi-free automorphisms* Operator algebras and their applications (Waterloo, ON, 1994/1995), 173–192.
- A. Kumjian, D. Pask, C^*-algebras of directed graphs and group actions. Ergodic Theory Dynam. Systems 19 (1999), no. 6, 1503–1519.
- M. Pimsner, *A Class of C*-Algebras Generalizing both Cuntz-Krieger Algebras and Crossed Products by* $\mathbb{Z}$.
- M. Rieffel, *Proper actions of groups on* C^**-algebras*, Mappings of operator algebras (Philadelphia, PA, 1988), 141–182.
- M. Rieffel, *Integrable and proper actions on* C^**-algebras, and square-integrable representations of groups*, Expo. Math. 22 (2004) 1–53.