Group actions on graphs and Doplicher-Roberts algebras

Valentin Deaconu, Alex Kumjian

(preliminary version)

Groupoidfest, San Bernardino, November 10, 2012
Let a group $G$ act on a directed graph $E$. This determines a representation $\rho$ of $G$ on the $C^*$-correspondence $\mathcal{H}_E$ and an action on $C^*(E)$.

Our goal is to study the crossed product $C^*(E) \rtimes G$ and the fixed point algebra $C^*(E)^G$ when $G$ is compact and the action is arbitrary.

We define the Doplicher-Roberts algebra $\mathcal{O}_\rho$ associated to $\rho$, constructed from intertwiners $(\rho^m, \rho^n)$, where $\rho^n = \rho \otimes^n$ on $\mathcal{H}_E^\otimes^n$.

In some cases, $\mathcal{O}_\rho \cong C^*(E)^G$ is strongly Morita equivalent to $C^*(E) \rtimes G$ and their K-theory can be computed.

We discuss the crossed product of a $C^*$-correspondence by a group $G$.

In particular we prove that $C^*(E) \rtimes G$ is SME to a graph algebra if $E$ and $G$ are finite.
Let $E = (E^0, E^1, r, s)$ be a topological graph. Denote by $\mathcal{H} = \mathcal{H}_E$ its $C^*$-correspondence over $A = C_0(E^0)$. A locally compact group $G$ acts on $E$ if there is a continuous morphism $G \to \text{Aut}(E)$.

We get a representation $\rho : G \to \mathcal{L}_\mathbb{C}(\mathcal{H})$ by invertible $\mathbb{C}$-linear operators on $\mathcal{H}$ and an action of $G$ on $A$ by $\ast$-automorphisms such that

$$\langle \rho(g)(\xi), \rho(g)(\eta) \rangle = g \cdot \langle \xi, \eta \rangle,$$

$$\rho(g)(\xi a) = (\rho(g)(\xi))(g \cdot a), \quad \rho(g)(a \cdot \xi) = (g \cdot a)(\rho(g)(\xi)).$$

We also get an action of $G$ on the Cuntz-Pimsner algebra $C^*(E)$ associated to the $C^*$-correspondence $\mathcal{H}$. 
If $G, E$ are discrete, the action is free and $E$ is locally finite, inspired from a theorem of Green, Kumjian and Pask proved in [7] that

$$C^*(E) \rtimes G \cong C^*(E/G) \otimes K(\ell^2(G)).$$

They also proved that if $G$ is abelian, and $c : E^1 \to \hat{G}$ is a cocycle, then $C^*(E) \rtimes G$ is isomorphic to $C^*(E(c))$, where $E(c)$ is the skew product graph $(\hat{G} \times E^0, \hat{G} \times E^1, r, s)$ with

$$r(\chi, e) = (\chi c(e), r(e)), s(\chi, e) = (\chi, s(e)).$$

The action $\alpha$ of $G$ on $C^*(E)$ is given by $\alpha_g(S_e) = \langle c(e), g \rangle S_e$, where $S_e$ are the generators of $C^*(E)$.

If $G$ abelian acts on the $O_n$-graph with $E^1 = \{e_1, e_2, \ldots, e_n\}$ and $E^0 = \{v\}$, a cocycle $c : E^1 \to \hat{G}$ determines a representation $\rho$ of $G$ on $\mathcal{H} = \text{span}\{\xi_1, \xi_2, \ldots, \xi_n\}$, where $\rho(g)\xi_i = \langle c(e_i), g \rangle \xi_i$.

Conversely, a representation of $G$ of dimension $n$ determines a cocycle on the $O_n$-graph with values in $\hat{G}$. 
Consider $\rho^n : G \to \mathcal{L}_{\mathbb{C}}(\mathcal{H}^\otimes n)$, and let
\[(\rho^m, \rho^n) = \{ T : \mathcal{H}^\otimes n \to \mathcal{H}^\otimes m \mid T \text{ is } A\text{-bilinear and } T \rho^n = \rho^m T \}.
\]
It follows that the linear span of $\cup_{m,n}(\rho^m, \rho^n)$ has a natural multiplication and involution, after identifying $T$ with $T \otimes I$.

The Doplicher-Roberts algebra $\mathcal{O}_\rho$ is defined as the $C^*$-closure of the linear span of $\cup_{m,n}(\rho^m, \rho^n)$.

**Theorem.** Let $E$ be a topological graph such that $\mathcal{H}_E$ is finite projective and the left multiplication is injective. If $G$ is a compact group acting on $E$ and the representation $\rho$ on $\mathcal{H}_E$ is faithful, then $\mathcal{O}_\rho \cong C^*(E)^G$.

**Proof.** Since $\mathcal{H}_E$ is finite projective and the left multiplication is injective, $C^*(E)$ is isomorphic to the $C^*$-algebra generated by the span of $\cup_{m,n}\mathcal{L}(\mathcal{H}^\otimes n, \mathcal{H}^\otimes m)$ after we identify $T$ with $T \otimes I$.

Note that $G$ acts on $\mathcal{L}(\mathcal{H}^\otimes n, \mathcal{H}^\otimes m)$ by $(g \cdot T)(\xi) = \rho^m(g)T(\rho^n(g^{-1})\xi)$ and the fixed point algebra is $(\rho^m, \rho^n)$, so $C^*(E)^G$ is isomorphic to $\mathcal{O}_\rho$. 
**Corollary.** Moreover, if $C^*(E)$ is simple, $G$ is finite and the action is outer, then $\mathcal{O}_\rho$ and $C^*(E) \rtimes G$ are simple and have the same $K$-theory.

**Proof.** We use the following result of Kishimoto: if $A$ is simple and $\alpha : G \to Aut(A)$ is an outer action, then $A \rtimes_\alpha G$ is simple.

**Example.** Consider a finite group $G$ acting on the graph $E$ with one vertex and $n \geq 2$ edges. We denote by $\rho$ the corresponding representation on $\mathcal{H} = \mathcal{H}_E = \mathbb{C}^n$.

Let $\hat{G}$ denote the set of equivalence classes of irreducible unitary representations, and consider as in [9] a graph with the incidence matrix $B = B(\rho)$, where $B(v, w)$ is the multiplicity of $w$ in $v \otimes \rho$ for $v, w \in \hat{G}$.

It turns out that $\mathcal{O}_\rho$ is a full corner in the Cuntz-Krieger algebra $\mathcal{O}_B$.

More generally, given a faithful representation $\rho$ of a compact group, Kumjian et. al. (see [6]) realize $\mathcal{O}_\rho$ as a corner in a graph $C^*$-algebra and as a groupoid algebra.
For $G = S_n$, we get an outer action on the Cuntz algebra $\mathcal{O}_n$ such that $\mathcal{O}_n \rtimes S_n$ is simple and strongly Morita equivalent to $\mathcal{O}_\rho \cong \mathcal{O}_n^{S_n}$. In particular they have the same $K$-theory.

For $n = 3$, using the character table of $S_3$, it was calculated in [9] that

$$B = \begin{pmatrix}
1 & 1 & 0 \\
1 & 2 & 1 \\
0 & 1 & 1 \\
\end{pmatrix},$$

which gives

$$K_0(\mathcal{O}_3 \rtimes S_3) = K_0(\mathcal{O}_\rho) = K_0(\mathcal{O}_B) \cong \mathbb{Z},$$

$$K_1(\mathcal{O}_3 \rtimes S_3) = K_1(\mathcal{O}_\rho) = K_1(\mathcal{O}_B) \cong \mathbb{Z}.$$

In particular the action does not have the Rokhlin property (see [4]) since the $K$-theory maps induced by the inclusion $\mathcal{O}_3^{S_3} \hookrightarrow \mathcal{O}_3$ are not injective.
Crossed products

Let $A$ be a $C^*$-algebra and let $\mathcal{H}$ be a $C^*$-correspondence over $A$. An action of $G$ on $\mathcal{H}$ determines an action on the Cuntz-Pimsner algebra $\mathcal{O}_\mathcal{H}$, called quasi-free.

The crossed product $\mathcal{H} \rtimes G$ is defined by

$$\mathcal{H} \rtimes G = \mathcal{H} \otimes_\varphi (A \rtimes G),$$

where $\varphi : A \to \mathcal{L}(A \rtimes G)$ is the embedding of $A$ in the multiplier algebra of $A \rtimes G$, regarded as a Hilbert module over itself.

The crossed product $\mathcal{H} \rtimes G$ becomes a $C^*$-correspondence over $A \rtimes G$ after the completion of $C_c(G, \mathcal{H})$ using the operations

$$\langle \xi, \eta \rangle (t) = \int_G s^{-1} \cdot \langle \xi(s), \eta(st) \rangle ds,$$

$$(\xi \cdot f)(t) = \int_G \xi(s)s \cdot (f(s^{-1}t)) ds, \quad (f \cdot \xi)(t) = \int_G f(s) \cdot (s \cdot \xi(s^{-1}t)) ds,$$

where $\xi \in C_c(G, \mathcal{H}), f \in C_c(G, A)$. 

Valentin Deaconu, Alex Kumjian
The right and left multiplications are just convolution. The inner product could be also expressed as

\[ \langle \xi \otimes f, \eta \otimes f' \rangle = f^* \langle \xi, \eta \rangle f', \]

where \( \xi, \eta \in \mathcal{H}, f, f' \in C_c(G, A) \).

Hao and Ng in [3] proved that if \( G \) is amenable, then

\[ \mathcal{O}_{\mathcal{H} \rtimes G} \cong \mathcal{O}_{\mathcal{H}} \rtimes G. \]

**Corollary.** If a finite group \( G \) acts on a finite graph \( E \), then the study of the crossed product \( C^*(E) \rtimes G \) reduces to the study of \( C^* \)-algebras arising from a finite dimensional \( C^* \)-correspondence \( \mathcal{H}_E \rtimes G \) over a finite dimensional algebra \( C(E^0) \rtimes G \).
Finite dimensional $C^*$-correspondences

Since any finite dimensional $C^*$-algebra is SME to $\mathbb{C}^n$ for some $n$, we first study the case when $C(E^0) \rtimes G$ is abelian. Suppose $A = C(V)$ with $V$ finite, and denote by $\{p_t\}_{t \in V}$ the minimal projections.

**Proposition.** The isomorphism classes of finitely generated $C^*$-correspondences $\mathcal{H}$ over $A = C(V)$ with unital $\ast$-homomorphisms $\varphi : A \to \mathcal{L}(\mathcal{H})$ correspond to matrices $(a_{st})_{s,t \in V}$ with nonnegative integer entries, having no row or column equal to zero. More precisely, $a_{st} = \dim \varphi(p_s)\mathcal{H}p_t$.

**Corollary.** If $C(E^0) \rtimes G = C(V)$ is abelian, then $C^*(E) \rtimes G$ is the graph algebra with incidence matrix $(a_{st})_{s,t \in V}$.

It is also the $C^*$-algebra of a graph of $C^*$-correspondences where the vertex algebras are $\mathbb{C}$ (one for each vertex $t \in V$) and the $C^*$-correspondences are finite dimensional Hilbert spaces of dimension $a_{st}$.
• **Theorem.** Suppose $A$ and $B$ are SME $C^*$-algebras with $A$-$B$ imprimitivity bimodule $\mathcal{X}$. If $\mathcal{H}$ is a $C^*$-correspondence over $A$, then $\mathcal{H}' = \mathcal{X}^* \otimes_A \mathcal{H} \otimes_A \mathcal{X}$ is a $C^*$-correspondence over $B$ such that $\mathcal{O}_\mathcal{H}$ and $\mathcal{O}_{\mathcal{H}'}$ are SME.

• **Proof.** Let $\mathcal{R} = \mathcal{H} \otimes_A \mathcal{X}$ and let $S = \mathcal{X}^*$. Then $\mathcal{R} \otimes_B S \cong \mathcal{H}, S \otimes_A \mathcal{R} \cong \mathcal{H}'$, so by a theorem in [8] (see also [10]), we get that $\mathcal{O}_\mathcal{H}$ and $\mathcal{O}_{\mathcal{H}'}$ are SME.

• **Corollary.** Given a finite graph $E = (E^0, E^1, r, s)$ and a finite group $G$ acting on $E$, the crossed product $C^*(E) \rtimes G$ is SME to a graph $C^*$-algebra, where the number of vertices is the cardinality of the spectrum of $C(E^0) \rtimes G$. 

Valentin Deaconu, Alex Kumjian  
Group actions on graphs and Doplicher-Roberts algebras
Examples

- Let $\mathbb{Z}_n$ act freely on its cyclic Cayley graph $E$ with $n$ vertices and $n$ edges.
- We get a representation $\rho$ of $\mathbb{Z}_n$ on $\mathcal{H} = \mathbb{C}^n$ and an action of $\mathbb{Z}_n$ on $A = \mathbb{C}^n$ which permutes cyclically the basis.
- We have $\mathcal{L}_A(\mathcal{H}) = \{ T \in M_n : T(\xi a) = T(\xi)a \} \cong \mathbb{C}^n$ (diagonal matrices).
- Moreover, $(\rho, \rho) \cong \mathbb{C}I$ and $\mathcal{O}_\rho \cong C^*(E/\mathbb{Z}_n) \cong C(\mathbb{T})$.
- The crossed product $\mathcal{H} \rtimes \mathbb{Z}_n \cong M_n$ becomes a $C^*$-correspondence over $\mathbb{C}^n \rtimes \mathbb{Z}_n \cong M_n$, and $\mathcal{O}_{\mathcal{H} \rtimes \mathbb{Z}_n} \cong M_n \otimes C(\mathbb{T})$. 
Examples

- Consider the graph $E$

![Graph Image]

- We have $C^*(E) \cong M_2 \otimes \mathcal{O}_2 \cong \mathcal{O}_2$.
- Let $G = \mathbb{Z}_2$ act on $E$ such that it permutes vertices and pairs of edges (free action). This gives a representation $\rho$ of $\mathbb{Z}_2$ on $\mathbb{C}^4$.
- It follows that $\mathcal{O}_\rho \cong C^*(E)\mathbb{Z}_2 \cong C^*(E/\mathbb{Z}_2) \cong \mathcal{O}_2$.
- Here $A = C(E^0) = \mathbb{C}^2$ and $\mathcal{H} = \mathbb{C}^4$ with corresponding bimodule structure and inner product.
- Then $\mathcal{H} \rtimes G \cong M_2(\mathbb{C}^2)$, a $C^*$-correspondence over $A \rtimes G \cong M_2$.
- Moreover, $\mathcal{O}_{\mathcal{H} \rtimes G} \cong M_2(\mathcal{O}_2) \cong \mathcal{O}_2$.
Examples

- Let $S_3$ act on the graph $E$ with one vertex and three loops by permuting the loops. We get a non-free action of $S_3$ on $O_3$.

- Since $S_3 \cong \mathbb{Z}_3 \rtimes \mathbb{Z}_2$, we have $O_3 \rtimes S_3 \cong (O_3 \rtimes \mathbb{Z}_3) \rtimes \mathbb{Z}_2$.

- It follows that $O_3 \rtimes \mathbb{Z}_3$ is isomorphic to $C^*(E(c))$, where $c : E^1 \to \widehat{\mathbb{Z}_3}$ is a cocycle and $E(c)$ is the graph with three vertices $v_1, v_2, v_3$ and nine edges connecting each $v_i$ with $v_j$.

- The group $\mathbb{Z}_2$ acts on $E(c)$ by fixing $v_1$ and interchanging $v_2$ with $v_3$. Let $\rho$ be the corresponding representation on $\mathbb{C}^9$.

- We have $C^*(E(c)) \cong O_3$, so we get a non-free action of $\mathbb{Z}_2$ on $O_3$. Here $A = \mathbb{C}^3$ and $A \rtimes \mathbb{Z}_2 \cong \mathbb{C}^2 \oplus M_2$. We get the $C^*$-correspondence $\mathbb{C}^9 \rtimes \mathbb{Z}_2 = \mathbb{C}^9 \otimes_{\mathbb{C}^3} (\mathbb{C}^2 \oplus M_2)$ over $\mathbb{C}^2 \oplus M_2$.

- It follows that $O_\rho \cong C^*(E(c))^{\mathbb{Z}_2}$ and $O_3 \rtimes S_3 \cong C^*(E(c)) \rtimes \mathbb{Z}_2$ is the graph algebra determined by a $3 \times 3$ matrix.


