Group actions on graphs

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(preliminary version)

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Let a group $G$ act on a directed graph $E$. This determines a representation $\rho$ of $G$ on the $C^*$-correspondence $\mathcal{H}_E$ and an action on the graph algebra $C^*(E)$.

Our goal is to study the fixed point algebra $C^*(E)^G$ and the crossed product $C^*(E) \rtimes G$ when $G$ is compact.

We define the Doplicher-Roberts algebra $O_\rho$ associated to $\rho$, constructed from intertwiners $(\rho^m, \rho^n)$, where $\rho^n = \rho \otimes^n$ on $\mathcal{H}_E^\otimes n$.

If $\mathcal{H}_E$ is finite projective, we prove that $O_\rho \cong C^*(E)^G$.

If $E$ and $G$ are finite, we prove that $C^*(E) \rtimes G$ is isomorphic to the $C^*$-algebra of a graph of (minimal) $C^*$-correspondences and is SME to a graph algebra.

This gives a method of computing the $K$-theory of $C^*(E) \rtimes G$. 
Let $E = (E^0, E^1, r, s)$ be a (topological) graph. Denote by $\mathcal{H} = \mathcal{H}_E$ its $C^*$-correspondence over $A = C_0(E^0)$, constructed from $C_c(E^1)$.

A locally compact group $G$ acts on $E$ if there is a continuous morphism $G \rightarrow Aut(E)$.

We get a representation $\rho : G \rightarrow \mathcal{L}_\mathbb{C}(\mathcal{H})$ by invertible $\mathbb{C}$-linear operators on $\mathcal{H}$ and an action of $G$ on $A$ by $\ast$-automorphisms such that

$$\langle \rho(g)(\xi), \rho(g)(\eta) \rangle = g \cdot \langle \xi, \eta \rangle,$$

$$\rho(g)(\xi a) = (\rho(g)(\xi))(g \cdot a), \quad \rho(g)(a \cdot \xi) = (g \cdot a)(\rho(g)(\xi)).$$

We also get an action of $G$ on the graph algebra $C^*(E)$ associated to the $C^*$-correspondence $\mathcal{H} = \mathcal{H}_E$. 
If $G, E$ are discrete, the action is free and $E$ is locally finite, Kumjian and Pask proved that
\[ C^* (E)^G \cong C^* (E/G), \]
\[ C^* (E) \rtimes G \cong C^* (E/G) \otimes K(\ell^2 (G)). \]

A similar result for the reduced crossed product $C^* (E) \rtimes_r G$ was proved by D., Kumjian and Quigg for free and proper actions of locally compact groups on topological graphs.

If $G$ is abelian and the action of $G$ on $C^* (E)$ is associated to a cocycle $c : E^1 \to \hat{G}$, then
\[ C^* (E) \rtimes G \cong C^* (E(c)), \]
where $E(c)$ is the skew product graph $(\hat{G} \times E^0, \hat{G} \times E^1, r, s)$ with
\[ r(\chi, e) = (\chi c(e), r(e)), s(\chi, e) = (\chi, s(e)). \]
Doplicher-Roberts algebras

- If $G$ acts on a $A$–$A$ $C^*$-correspondence $\mathcal{H}$ via $\rho : G \to \mathcal{L}_\mathbb{C}(\mathcal{H})$, consider the tensor power $\rho^n : G \to \mathcal{L}_\mathbb{C}(\mathcal{H}^\otimes n)$ and 
\[(\rho^m, \rho^n) = \{T : \mathcal{H}^\otimes n \to \mathcal{H}^\otimes m \mid T \text{ is } A\text{-linear and } T \rho^n = \rho^m T\}\].

- The linear span of $\bigcup_{m,n}(\rho^m, \rho^n)$ has a natural multiplication and involution, after identifying $T$ with $T \otimes I$.

- The Doplicher-Roberts algebra $\mathcal{O}_\rho$ is defined as the $C^*$-closure of the linear span of $\bigcup_{m,n}(\rho^m, \rho^n)$.

**Theorem.** Let $E$ be a topological graph such that $\mathcal{H}_E$ is finite projective and the left multiplication is injective. If $G$ is a compact group acting on $E$, then $\mathcal{O}_\rho \cong C^*(E)^G$.

**Proof.** Since $\mathcal{H}_E$ is finite projective and the left multiplication is injective, $C^*(E)$ is isomorphic to the $C^*$-algebra generated by the span of $\bigcup_{m,n}\mathcal{L}(\mathcal{H}^\otimes n, \mathcal{H}^\otimes m)$ after we identify $T$ with $T \otimes I$.

- Note that $G$ acts on $\mathcal{L}(\mathcal{H}^\otimes n, \mathcal{H}^\otimes m)$ by $(g \cdot T)(\xi) = \rho^m(g)T(\rho^n(g^{-1})\xi)$ and the fixed point algebra is $(\rho^m, \rho^n)$, so $C^*(E)^G$ is isomorphic to $\mathcal{O}_\rho$. 

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Corollary. Moreover, if $C^*(E)$ is simple, $G$ is finite and the action is outer, then $O_\rho$ and $C^*(E) \rtimes G$ are simple and have the same $K$-theory.

Proof. We use the following result of Kishimoto: if $A$ is simple and $\alpha : G \to Aut(A)$ is an outer action, then $A \rtimes_\alpha G$ is simple.

Example. Let a finite group $G$ act on the graph $E$ with one vertex and $n \geq 2$ edges. Denote by $\rho$ the corresponding representation on $\mathcal{H} = \mathcal{H}_E = \mathbb{C}^n$.

Consider the graph with the incidence matrix $B = B(\rho)$, where $B(v, w)$ is the multiplicity of $w$ in $v \otimes \rho$ for $v, w \in \hat{G}$.

Raeburn et.al. proved that $O_\rho$ is a full corner in the Cuntz-Krieger algebra $O_B$.

More generally, given a faithful representation $\rho$ of a compact group, Kumjian et. al. realize $O_\rho$ as a corner in a graph $C^*$-algebra and as a groupoid algebra.
Crossed products of $C^*$-correspondences

- Let $A$ be a $C^*$-algebra and let $\mathcal{H}$ be a $C^*$-correspondence over $A$. An action of $G$ on $\mathcal{H}$ and $A$ as above determines an action on the Cuntz-Pimsner algebra $\mathcal{O}_\mathcal{H}$.

- The crossed product $\mathcal{H} \rtimes G = \mathcal{H} \otimes_A (A \rtimes G)$ becomes a $C^*$-correspondence over $A \rtimes G$ after the completion of $C_c(G, \mathcal{H})$ using the operations

$$\langle \xi, \eta \rangle(t) = \int_G s^{-1} \cdot \langle \xi(s), \eta(st) \rangle ds,$$

$$\langle \xi \cdot f \rangle(t) = \int_G \xi(s) s \cdot (f(s^{-1}t)) ds, \quad (f \cdot \xi)(t) = \int_G f(s) \cdot (s \cdot \xi(s^{-1}t)) ds,$$

where $\xi \in C_c(G, \mathcal{H}), f \in C_c(G, A)$.

- Hao and Ng proved that if $G$ is amenable, then

$$\mathcal{O}_{\mathcal{H} \rtimes G} \cong \mathcal{O}_\mathcal{H} \rtimes G.$$
Main result

- **Theorem.** Given a finite graph $E$ and a finite group $G$ acting on $E$, the crossed product $C^*(E) times G$ is the $C^*$-algebra of a graph of $C^*$-correspondences. Moreover, $C^*(E) times G$ is SME to a graph $C^*$-algebra, where the number of vertices is the cardinality of the spectrum of $C(E^0) times G$.

- **Proof.** We study the finite dimensional $C^*$-correspondence $\mathcal{H}_E times G$ over the finite dimensional $C^*$-algebra $C(E^0) times G$.

Let $C(E^0) times G \cong \bigoplus_{i=1}^{n} A_i$, where $A_i$ are matrix algebras. This decomposition is obtained in two stages, from the orbits in $E^0$ and from the characters of the stabilizer groups.

Consider the graph with $n$ vertices and at each vertex $v_i$ we assign the $C^*$-algebra $A_i$. The edges and the assigned $C^*$-correspondences are constructed from the orbits in $E^1$ and multiplicities.
It follows that $C^*(E) \rtimes G$ is isomorphic to the $C^*$-algebra of this graph of (minimal) $C^*$-correspondences.

For the second part we use

**Lemma.** Suppose $A$ and $B$ are SME $C^*$-algebras with $A$-$B$ imprimitivity bimodule $\mathcal{X}$. If $\mathcal{H}$ is a $C^*$-correspondence over $A$, then $\mathcal{H}' = \mathcal{X}^* \otimes_A \mathcal{H} \otimes_A \mathcal{X}$ is a $C^*$-correspondence over $B$ such that $\mathcal{O}_\mathcal{H}$ and $\mathcal{O}_{\mathcal{H}'}$ are SME.

**Proof.** Let $\mathcal{R} = \mathcal{H} \otimes_A \mathcal{X}$ and let $S = \mathcal{X}^*$. Then $\mathcal{R} \otimes_B S \cong \mathcal{H}, S \otimes_A \mathcal{R} \cong \mathcal{H}'$, so by a theorem of Muhly and Solel, we get that $\mathcal{O}_\mathcal{H}$ and $\mathcal{O}_{\mathcal{H}'}$ are SME.

**Corollary.** The $K$-theory of $C^*(E) \rtimes G$ can be computed as the $K$-theory of a graph algebra.
Let $S_3$ act on the graph $E$ with one vertex and three loops by permuting the loops. We get a representation $\rho : S_3 \rightarrow \mathcal{L}(\mathbb{C}^3)$ and an action on $\mathcal{O}_3$.

Since $S_3 \cong \mathbb{Z}_3 \rtimes \mathbb{Z}_2$, we have $\mathcal{O}_3 \rtimes S_3 \cong (\mathcal{O}_3 \rtimes \mathbb{Z}_3) \rtimes \mathbb{Z}_2$.

It follows that $\mathcal{O}_3 \rtimes \mathbb{Z}_3$ is isomorphic to $C^*(E(c))$, where $c : E^1 \rightarrow \hat{\mathbb{Z}}_3$ is a cocycle and $E(c)$ is the graph with three vertices $v_1, v_2, v_3$ and nine edges connecting each $v_i$ with $v_j$.

The group $\mathbb{Z}_2$ acts on $E(c)$ by fixing $v_1$ and interchanging $v_2$ with $v_3$. Here $A = \mathbb{C}^3$, $A \rtimes \mathbb{Z}_2 \cong \mathbb{C} \oplus \mathbb{C} \oplus M_2 \cong C^*(S_3)$, $\mathcal{H}_{E(c)} = \mathbb{C}^9$ and $\mathcal{H}_{E(c)} \rtimes \mathbb{Z}_2$ decomposes accordingly.

It follows that $\mathcal{O}_\rho \cong \mathcal{O}_3^{S_3}$ and $\mathcal{O}_3 \rtimes S_3$ are SME to the graph algebra with incidence matrix

$$
\begin{pmatrix}
1 & 0 & 1 \\
0 & 1 & 1 \\
1 & 1 & 2
\end{pmatrix}
$$

Therefore,$$
K_0(\mathcal{O}_3 \rtimes S_3) \cong K_0(\mathcal{O}_\rho) \cong \mathbb{Z}, \ K_1(\mathcal{O}_3 \rtimes S_3) \cong K_1(\mathcal{O}_\rho) \cong \mathbb{Z}.
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References


