Group actions on graphs and Doplicher-Roberts algebras

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(preliminary report)

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Let the group $G$ act on a directed graph $E$. This determines a representation $\rho$ of $G$ on the $C^*$-correspondence $\mathcal{H}_E$ and an action on $C^*(E)$.

We define the Doplicher-Roberts algebra $\mathcal{O}_\rho$ associated to $\rho$, constructed from intertwiners $(\rho^m, \rho^n)$, where $\rho^n = \rho \otimes^n$ on $\mathcal{H}_E^n$.

Goal: to study the crossed product $C^*(E) \rtimes G$ and the fixed point algebra $C^*(E)^G$ when $G$ is compact and the action is not free.

In some cases, $\mathcal{O}_\rho$ is strongly Morita equivalent to $C^*(E) \rtimes G$ and their K-theory can be computed.

We also discuss the crossed product of a $C^*$-correspondence by a group $G$, with examples.
Let $E = (E^0, E^1, r, s)$ be a topological graph. Denote by $\mathcal{H} = \mathcal{H}_E$ its $C^*$-correspondence over $A = C_0(E^0)$. A locally compact group $G$ acts on $E$ if there is a continuous morphism $G \to Aut(E)$.

We get a representation $\rho : G \to \mathcal{L}_{\mathbb{C}}(\mathcal{H})$ by invertible $\mathbb{C}$-linear operators on $\mathcal{H}$ and an action of $G$ on $A$ by $*$-automorphisms such that

$$\langle \rho(g)(\xi), \rho(g)(\eta) \rangle = g \cdot \langle \xi, \eta \rangle,$$

$$\rho(g)(\xi a) = (\rho(g)(\xi))(g \cdot a), \quad \rho(g)(a \cdot \xi) = (g \cdot a)(\rho(g)(\xi)).$$

We also get an action on the Cuntz-Pimsner algebra (or graph algebra) $C^*(E)$ associated to the $C^*$-correspondence $\mathcal{H}$.

If $G, E$ are discrete, the action is free and $E$ is locally finite, inspired from a theorem of Green, Kumjian and Pask proved that

$$C^*(E) \rtimes G \cong C^*(E/G) \otimes \mathcal{K}(\ell^2(G)).$$

What about arbitrary actions?
Doplicher-Roberts algebras

Consider \( \rho^n : G \to \mathcal{L}_{\mathbb{C}}(\mathcal{H}^\otimes n) \), and let
\[
(\rho^m, \rho^n) = \{ T : \mathcal{H}^\otimes n \to \mathcal{H}^\otimes m \mid T \text{ is } A\text{-bilinear and } T\rho^m = \rho^m T \}.
\]

It follows that \( \bigcup_{m,n} (\rho^m, \rho^n) \) has a natural multiplication and involution, after identifying \( T \) with \( T \otimes I \).

We define the Doplicher-Roberts algebra \( \mathcal{O}_\rho \) as the \( C^* \)-closure of \( \bigcup_{m,n} (\rho^m, \rho^n) \).

**Theorem.** Let \( E \) be a topological graph such that \( \mathcal{H}_E \) is finite projective and the left multiplication is injective. If \( G \) is a compact group acting on \( E \) and the representation \( \rho \) on \( \mathcal{H}_E \) is faithful, then \( \mathcal{O}_\rho \cong C^* (E)^G \).

**Corollary.** Moreover, if \( C^* (E) \) is simple, \( G \) is finite and the action is outer, then \( \mathcal{O}_\rho \) and \( C^* (E) \rtimes G \) are simple and have the same \( K \)-theory.

We use the following theorem of A. Kishimoto: If \( A \) is simple and \( \alpha : G \to Aut(A) \) is an outer action, then \( A \rtimes_\alpha G \) is simple.
Consider a finite group $G$ acting on the graph $E$ with one vertex and $n \geq 2$ edges. We denote by $\rho$ the corresponding representation on $\mathcal{H} = \mathcal{H}_E = \mathbb{C}^n$.

Let $\hat{G}$ denote the set of equivalence classes of irreducible unitary representations, and consider as in [7] a graph with the incidence matrix $B = B(\rho)$, where $B(v, w)$ is the multiplicity of $w$ in $v \otimes \rho$ for $v, w \in \hat{G}$.

It turns out that $\mathcal{O}_\rho$ is a full corner in the Cuntz-Krieger algebra $\mathcal{O}_B$.

More generally, given a faithful representation $\rho$ of a compact group, Kumjian et. al. (see [5]) realize $\mathcal{O}_\rho$ as a corner in a graph $C^*$-algebra and as a groupoid algebra.
• For $G = S_n$, we get an outer action on the Cuntz algebra $O_n$ such that $O_n \rtimes S_n$ is simple and strongly Morita equivalent to $O_\rho \cong O_n^{S_n}$. In particular they have the same $K$-theory.

• For $n = 3$, using the character table of $S_3$, it was calculated in [7] that

$$B = \begin{pmatrix}
1 & 1 & 0 \\
1 & 2 & 1 \\
0 & 1 & 1
\end{pmatrix},$$

which gives

$$K_0(O_3 \rtimes S_3) = K_0(O_\rho) = K_0(O_B) \cong \mathbb{Z},$$

$$K_1(O_3 \rtimes S_3) = K_1(O_\rho) = K_1(O_B) \cong \mathbb{Z}.$$  

• In particular the action does not have the Rokhlin property.
An action of $G$ on a $C^*$-correspondence $\mathcal{H}$ over $A$ determines an action on the Cuntz-Pimsner algebra $O_{\mathcal{H}}$, called quasi-free.

The crossed product $\mathcal{H} \rtimes G$ can be defined by $\mathcal{H} \rtimes G = \mathcal{H} \otimes_\varphi (A \rtimes G)$, where $\varphi : A \to \mathcal{L}(A \rtimes G)$ is the embedding of $A$ in the multiplier algebra of $A \rtimes G$, regarded as a Hilbert module over itself.

The crossed product $\mathcal{H} \rtimes G$ becomes a $C^*$-correspondence over $A \rtimes G$ in a natural way.

It is proved in [3, Theorem 2.10] that if $G$ is amenable, then

$$O_{\mathcal{H} \rtimes G} \cong O_{\mathcal{H}} \rtimes G.$$
Examples

- Consider the graph $E$

  \[ \begin{array}{c}
  e_1 \\
  \vdots \\
  e_3 \\
  \end{array} \quad \begin{array}{c}
  v_1 \\
  \vdots \\
  v_2 \\
  \end{array} \quad \begin{array}{c}
  e_2 \\
  \vdots \\
  e_4 \\
  \end{array} \]

- We have $C^*(E) \cong M_2 \otimes \mathcal{O}_2 \cong \mathcal{O}_2$.
- Let $G = \mathbb{Z}_2$ act on $E$ such that it permutes vertices and pairs of edges (free action). It follows that $E/G$ has one vertex and two loops, so $C^*(E/G) \cong \mathcal{O}_2$.
- We have $A = \mathbb{C}^2$ and $\mathcal{H} = \mathbb{C}^4$ with corresponding bimodule structure and inner product.
- Then $\mathcal{H} \rtimes G \cong M_2(\mathbb{C}^2)$, a $C^*$-correspondence over $A \rtimes G \cong M_2$. Moreover, $\mathcal{O}_{\mathcal{H} \rtimes G} \cong M_2(\mathcal{O}_2) \cong \mathcal{O}_2$. 
Examples

- Let $\mathbb{Z}_n$ act freely on its cyclic Cayley graph $E$.
- We get a representation $\rho$ of $\mathbb{Z}_n$ on $\mathcal{H} = \mathbb{C}^n$ and an action of $\mathbb{Z}_n$ on $A = \mathbb{C}^n$ which permutes cyclically the basis.
- We have $\mathcal{L}_A(\mathcal{H}) = \{ T \in M_n : T(\xi a) = T(\xi) a \} \cong \mathbb{C}^n$ (diagonal matrices).
- Moreover, $(\rho, \rho) \cong \mathbb{C} I$ and $\mathcal{O}_\rho \cong C(\mathbb{T})$.
- The crossed product $\mathcal{H} \rtimes \mathbb{Z}_n \cong M_n$ becomes a $C^*$-correspondence over $M_n$, and $\mathcal{O}_{\mathcal{H} \rtimes \mathbb{Z}_n} \cong M_n \otimes C(\mathbb{T})$. 
Let again $S_n$ act on the graph with one vertex and $n$ loops by permuting the loops (non-free action).

Here $A = \mathbb{C}$, $\mathcal{H} = \mathbb{C}^n$ and $\mathcal{H} \rtimes S_n \cong \mathbb{C}^n \otimes C^*(S_n)$ becomes a $C^*$-correspondence over $C^*(S_n)$.

We have $\mathcal{O}_{\mathcal{H} \rtimes S_n} \cong \mathcal{O}_n \rtimes S_n$, which gives a new perspective on the crossed product.

Question: What $C^*$-algebras arise from a finite dimensional $C^*$-correspondence over a finite dimensional algebra?


