Groups and Geometry

Valentin Deaconu

Undergraduate Colloquium, Reno, 15 March 2007

An influential research program and manifesto was published in 1872 by the german mathematician Felix Klein, known as the **Erlangen Program**. This proposed that group theory, an algebraic approach that encapsulates the idea of symmetry, was the correct way of organizing geometrical knowledge. For example, plane euclidean geometry is associated with the group of transformation of the plane that preserve the euclidean distance. Plane projective geometry is associated with the group of projective transformations. Plane hyperbolic geometry can be associated with the group of projective transformations that map the unit circle onto itself.

Groups

Definition. A group is a set G with an associative binary operation, an identity element, and inverses.

Examples. 1) The cyclic group $\mathbb{Z}/n\mathbb{Z}$ of classes modulo n.

2) The group S_3 of permutations of $\{1, 2, 3\}$, or the symmetries of an equilateral triangle. More general we have S_n.

3) The dihedral group D_n of symmetries of a regular n-gon. Note that $D_n \subset S_n$.

4) The linear group

$$GL(2, F) = \{A \in M_2(F) \mid \ A \text{ invertible}\},$$

where F is $\mathbb{R}$ or $\mathbb{C}$. It has a (normal) subgroup

$$SL(2, F) = \{A \in M_2(F) \mid \det(A) = 1\}.$$

5) The Moebius group $M = PGL(2, \mathbb{C}) = PSL(2, \mathbb{C}) = SL(2, \mathbb{C})/\{\pm I\}$.

6) The orthogonal group $O(2) = \{V \in M_2(\mathbb{R}) \mid VV^t = I\}$. It has a subgroup $SO(2) = \{V \in O(2) \mid \det(V) = 1\}$ (of rotations).

7) The affine group $AGL(2, F) = \{T_{A,b} \mid A \in GL(2, F), b \in F^2\}$, where $T_{A,b}(x) = Ax + b$. It takes lines into lines.

8) The Euclidean group $Iso(\mathbb{E}^2) = \{T_{A,b} \mid A \in O(2), b \in \mathbb{R}^2\}$.

The Euclidean plane

Take $\mathbb{E}^2 = \mathbb{R}^2$, with the distance

$$d(p,q) = |p-q| = \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}.$$

An isometry of $\mathbb{E}^2$ is a bijection $\sigma : \mathbb{E}^2 \to \mathbb{E}^2$ such that $d(\sigma(p), \sigma(q)) = d(p,q)$. Examples:

1. R^p_θ, the rotation with center p and angle θ.

2. T_v, the translation of vector v.

3. R_ℓ, the reflection in a line ℓ. This reverses the orientation. Note that a rotation or a translation is a composition of two reflections.

4. $G_{\ell,v}$, the glide reflection along a line ℓ with vector v parallel to ℓ. In fact $G_{\ell,v} = T_v \circ R_\ell = R_\ell \circ T_v$.

Any isometry of $\mathbb{E}^2$ is one of these geometric transformations.

For X a figure in the plane (any subset), we could construct the symmetry group

$$Symm(X) = \{S \in Iso(\mathbb{E}^2) \mid S(X) = X\}.$$

Theorem. For $n \geq 3$, the symmetry group $Symm(P_n)$ of the regular n-gon P_n consists of the rotations $R_{2k\pi/n}, k = 0, 1, ..., n-1$ and n reflections $R_{\ell_1}, ..., R_{\ell_n}$ in the lines $\ell_1, ..., \ell_n$ joining the origin to the vertices and to the midpoints of the sides.

Letting $g = R_{2\pi/n}$ and $h = R_{\ell_1}$, we have $h^2 = e$ and $hg = g^{-1}h$. The elements of $Symm(P_n)$ are

$$e, g, g^2, ..., g^{n-1}, h, gh, g^2h, ..., g^{n-1}h.$$

This group is the dihedral group D_n of order $2n$.

Theorem. Any finite subgroup $G \subset Iso(\mathbb{E}^2)$ fixes a point p_0 and is isomorphic to one of the following

1. The cyclic group of order n generated by $R^{p_0}_{2\pi/n}$

2. The dhedral group of order $2n$ generated by $R^{p_0}_{2\pi/n}$ and a reflection R_ℓ in a line ℓ through p_0.

Proof. Let p a point and consider the orbit

$$Gp = \{S(p) \mid S \in G\}$$

This is a finite set $\{p_1, ..., p_m\}$. Each element in G is a permutation of this set. Consider the centroid

$$p_0 = \frac{p_1 + ... + p_m}{m}$$

It follows that p_0 is left fixed.

Case 1. G contains only isometries preserving the orientation. Every element in G is a rotation with center p_0. Let θ the smallest angle of rotation. Then R_θ is the generator.

Case 2. G contains an isometry changing the orientation. This must be a reflection R_ℓ in a line ℓ through p_0.

Let G^+ be the subgroup of the isometries preserving the orientation. Then G^+ is generated by a rotation R_θ with $\theta = 2\pi/n$ and G is isomorphic to D_n since if $R_{\ell'}$ is another reflection in G, $R_{\ell'} \circ R_\ell$ is a rotation and is R_θ^k for some k.

Discrete subgroups of $Iso(\mathbb{E}^2)$

Let $G \subset Iso(\mathbb{E}^2)$ be a discrete group. If G contains a translation T_v with $v \neq 0$, then G is infinite. The same conclusion holds for glides, since the square of a glide is a translation.

Let T_G be the subgroup of translations in G. Since the translations in $Iso(\mathbb{E}^2)$ are isomorphic to $\mathbb{R}^2$, T_G is mapped to a discrete subgroup L_G of $\mathbb{R}^2$.

Then L_G is either trivial, either generated by a translation T_v and hence isomorphic to $\mathbb{Z}$ or generated by two translations T_v, T_w with v, w linearly independent, hence isomorphic to $\mathbb{Z}^2$.

In the first case G is finite, in the second case G is the symmetry group of a *frieze pattern*, and in the third case G is a *crystallographic group*. Considering the image of G in the orthogonal group $O(2)$, it turns out that there are 7 frieze groups and 17 crystallographic groups.

http://www.geom.uiuc.edu/java/Kali/program.html

Examples of tessellation

http://www.spsu.edu/math/tile/symm/types/index.htm

Complex numbers

A complex number is of the form $z = a + ib$, where $a, b \in \mathbb{R}$. The real part $Re(z)$ is a, and the imaginary part $Im(z)$ is b. We can write $z = r(\cos\theta + i\sin\theta)$, where $r = |z| = \sqrt{a^2 + b^2}$ and $\theta = \arg(z)$. The set of complex numbers $\mathbb{C}$ is in a one-to-one correspondence with $\mathbb{R}^2$. Operations:

$$z_1 + z_2 = (a_1 + a_2) + i(b_1 + b_2)$$

$$z_1 z_2 = (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1).$$

If we write $z_j = r_j(\cos\theta_j + i\sin\theta_j)$ for $j = 1, 2$, then

$$z_1 z_2 = r_1 r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)).$$

If $z = r(\cos\theta + i\sin\theta)$, then $\frac{1}{z} = \frac{1}{r}(\cos\theta - i\sin\theta)$.

Stereographic projection

Consider $\mathbb{C}$ imbedded in $\mathbb{R}^3$ by the map $z = a + ib \mapsto (a, b, 0)$. Let $S^2 = \{p \in \mathbb{R}^3 \mid |p| = 1\}$. If we sit at the North pole $N = (0, 0, 1)$ and look down to the Earth S^2, every time we see a location on the transparent Earth, we also see a unique point on $\mathbb{C}$. We get the stereographic projection $h_N : S^2 - \{N\} \to \mathbb{C}$. If we prefer the southern hemisphere, we get a map $h_S : S^2 - \{S\} \to \mathbb{C}$.

The composition $h_N \circ h_S^{-1} : \mathbb{C} - \{0\} \to \mathbb{C} - \{0\}$ is given by

$$h_N \circ h_S^{-1}(z) = \frac{z}{|z|^2},\ z \neq 0.$$

Using $\bar{z} = a - ib$ and $z\bar{z} = |z|^2$, we get

$$h_N \circ h_S^{-1}(z) = \frac{1}{\bar{z}}, z \neq 0.$$

Moebius geometry

Reflections in lines play a central role in Euclidean geometry. In fact, every isometry of $\mathbb{E}^2$ is the product of at most three reflections. In spherical geometry the space is S^2, lines are great circles and reflections are given by spatial reflections in planes of $\mathbb{E}^3$ spanned by great circles. For example, $h_S^{-1} \circ h_N : S^2 \to S^2$ is the reflection $(a, b, c) \mapsto (a, b, -c)$.

Using the stereographic projection $h_N : S^2 - \{N\} \to \mathbb{R}^2$, we can "view" all spherical objects in the plane. The spherical reflections like $z \mapsto 1/\bar{z}$ are singular when viewed on $\mathbb{R}^2$.

Let $\widehat{\mathbb{R}}^2 = \mathbb{R}^2 \cup \{\infty\}$ be the one point compactification. Then $z \mapsto 1/\bar{z}$ takes ∞ to 0.

We will consider all finite compositions of spherical reflections into great circles of S^2 and pull them down to $\widehat{\mathbb{R}}^2$ using the stereographic projection. We obtain the Moebius transformations.

In Moebius geometry we have to treat circles and lines on the same footing. When we talk about circles, we really mean circles or lines.

Define the reflection (inversion) $R_S : \widehat{\mathbb{R}}^2 \to \widehat{\mathbb{R}}^2$ in the circle

$$S = S_r(p_0) = \{p \in \mathbb{R}^2 \mid \ d(p, p_0) = r\}$$

by

$$R_S(p) = p_0 + \left(\frac{r}{d(p, p_0)}\right)^2 (p - p_0), p \in \widehat{\mathbb{R}}^2.$$

Note that $R_S(p_0) = \infty, R_S(\infty) = p_0$, R_S fixes the points on $S_r(p_0)$, and a computation shows that $R_S \circ R_S$ is the identity. Also, for $p_0 = 0$, we get $R_S(p) = p/|p|^2$.

For the reflection R_ℓ in the line $\ell_t(p_0) = \{p \in \mathbb{R}^2 \mid p \cdot p_0 = t\} \cup \{\infty\}$ with normal vector p_0, we have

$$R_\ell(p) = p - \frac{2(p \cdot p_0 - t)p_0}{|p_0|^2}, \; R_\ell(\infty) = \infty.$$

We define the Moebius group $Moe(\widehat{\mathbb{R}}^2)$ as the group of finite compositions of reflections into circles and lines. We have

$$Iso(\mathbb{E}^2) \subset Moe(\widehat{\mathbb{R}}^2),$$

by extending a plane isometry to $\widehat{\mathbb{R}}^2$ by declaring that they send ∞ to itself.

If we identify $\widehat{\mathbb{R}}^2$ with $\widehat{\mathbb{C}}$, let's express the basic geometric transformations in terms of operations with complex numbers

1. The translation $T_v(z) = z + v$.

2. The rotation R_θ with center at the origin $R_\theta(z) = z(\theta)z$. Rotation with center p_0 is $R_\theta^{p_0} = T_{p_0} \circ R_\theta \circ T_{p_0}^{-1}$, so

$$R_\theta^{p_0}(z) = z(\theta)(z - p_0) + p_0.$$

3. The reflection R_ℓ is given by

$$R_\ell(z) = -\frac{p_0\bar{z} - 2t}{\bar{p_0}}.$$

4. The glide $G_{\ell,v}$ has the formula

$$G_{\ell,v}(z) = T_v \circ R_\ell(z) = v - \frac{p_0\bar{z} - 2t}{\bar{p_0}}.$$

5. The reflection R_S into the circle $S_r(p_0)$ is given by

$$R_S(z) = \frac{r^2}{\bar{z} - \bar{p_0}} + p_0 = \frac{p_0\bar{z} + (r^2 - |p_0|^2)}{\bar{z} - \bar{p_0}}.$$

Theorem.The group $Moe^+(\widehat{\mathbb{C}})$ of orientation preserving Moebius transformations is equal to the group of complex linear fractional transformations

$$z \mapsto \frac{az+b}{cz+d},\ ad - bc \neq 0, a, b, c, d \in \mathbb{C}.$$

In fact, if $c = 0$, then $g(z) = (a/d)z + (b/d)$, so that g is a translation for $a = d$ and the composition of a translation, a rotation and a central dilation (composition of two reflections in circles centered at 0). If $c \neq 0$, then

$$g(z) = \frac{bc - ad}{c^2(z + d/c)} + \frac{a}{c},$$

hence g is the composition of a translation, the Moebius transformation $z \mapsto 1/z$, a rotation, a central dilation, and another translation.

We can assume $ad - bc = 1$, and there is an isomorphism

$$Moe^{+}(\widehat{\mathbb{C}}) \cong SL(2, \mathbb{C})/\{\pm I\}.$$

Note. The Moebius transformations which reverse the orientation can be written as

$$z \mapsto \frac{a\bar{z}+b}{c\bar{z}+d},\ ad-bc=1, a,b,c,d \in \mathbb{C}.$$

Hyperbolic geometry

The isometries of $\mathbb{E}^2$ map lines into lines. The Moebius transformations map circles (circles and lines) into circles. Let's find a geometry in which lines are circles and isometries are Moebius transformations. Let $\mathbb{H}^2$ to be the upper half-plane. Then through any two points p and q in $\mathbb{H}^2$, there is a unique "line" (a Euclidean half-line or a semicircle). The first four postulates of Euclid are satisfied, but the fifth fails. Recall

Postulate 1. For every points P and Q, $P \neq Q$ there exists a unique line ℓ passing through P and Q.

Postulate 2. For every segments AB and CD there exists a unique point E such that B is between A and E and CD is congruent with BE.

Postulate 3. For every points O and A with $O \neq A$ there exists a circle with center O and radius OA.

Postulate 4. All right angles are congruent to each other.

Postulate 5. For every line ℓ and every point P not on ℓ there exists a unique line m through P that is parallel to ℓ.

This new geometry was discovered about 1830 by Bolyai and Lobachevsky.

Theorem. A matrix in $SL(2,\mathbb{C})$ leaves $\mathbb{H}^2$ invariant iff its entries are real.

We see that a natural candidate for the group of transformations in hyperbolic geometry is $SL(2,\mathbb{R})$. The elements of $SL(2,\mathbb{R})$ preserve angles, but certainly do not preserve Euclidean distances. There is a natural distance function d_H with respect to which the elements of $SL(2,\mathbb{R})$ act as isometries.

$$d_H(z,w) = \cosh^{-1}(1 + \frac{|z-w|^2}{2Im(z)Im(w)}),$$

where $\cosh t = (e^t + e^{-t})/2$.

Fuchsian groups

A discrete subgroup of $SL(2,\mathbb{C})$ is called *Kleinian*. A typical example is $SL(2,\mathbb{Z})$. A Kleinian group that leaves a half-plane or a disc invariant is called *Fuchsian*. Fuchsian groups may be visualized as hyperbolic tilings or tessellations of $\mathbb{H}^2$.

Given a Fuchsian group $G \subset SL(2,\mathbb{R})$, we say that $F \subset \mathbb{H}^2$ is a *fundamental set* if F meets each orbit

$$Gz = \{g(z) \mid g \in G\}, z \in \mathbb{H}^2$$

exactly once. A *fundamental domain* for G is an open connected set $F_0 \subset \mathbb{H}^2$ such that there is a fundamental set F between F_0 and $\bar{F}_0$ and ∂F_0 has zero dimensional area.

Example 1. Let $g(z) = z + 1, z \in \mathbb{H}^2$. A fundamental domain F_0 for $G = \langle g \rangle$ is given by $0 < Re(z) < 1$.

Example 2. Let $k > 1$ and $g(z) = kz, z \in \mathbb{H}^2$. A fundamental domain for $G = \langle g \rangle$ is $F_0 = \{z \in \mathbb{H}^2 \mid 1 < |z| < k\}$

Example 3. Let $k > 1$ and $G = \langle g_1, g_2 \rangle$, where

$$g_1(z) = -1/z,\ g_2(z) = kz, z \in \mathbb{H}^2$$

A fundamental domain is

$$F_0 = \{z \in \mathbb{H}^2 \mid 1 < |z| < k, Re(z) > 0\}.$$

Example 4. Let $G = SL(2, \mathbb{Z})$ be the modular group. It is generated by $z \mapsto z + 1$ and $z \mapsto -1/z$. Then F_0 defined by $|z| > 1$ and $|Re(z)| < 1/2$ is a fundamental domain.

A group of isometries fo $\mathbb{H}^2$ is said to be a triangle group of type (α, β, γ) if G is generated by reflections in the sides of a hyperbolic triangle with angles α, β and γ.

Example 5. A hyperbolic tiling of a triangle group of type $(\pi/6, \pi/4, \pi/2)$

http://aleph0.clarku.edu/∼djoyce/poincare/poincare.html