4. Let $r_1, r_2, r_3 \in \mathbb{Q}$. We want to prove $r_1 + (r_2 + r_3) = (r_1 + r_2) + r_3$ and $r_1(r_2r_3) = (r_1r_2)r_3$. Let $r_i = \frac{a_i}{b_i}$ with $a_i, b_i \in \mathbb{Z}$ and $b_i \neq 0, i = 1, 2, 3$. Then

$$r_1 + (r_2 + r_3) = \frac{a_1}{b_1} + \left( \frac{a_2}{b_2} + \frac{a_3}{b_3} \right) = \frac{a_1}{b_1} + \frac{a_2b_3 + a_3b_2}{b_2b_3} = \frac{a_1b_2b_3 + (a_2b_3b_1 + a_3b_2b_1)}{b_1b_2b_3}$$

and

$$(r_1 + r_2) + r_3 = \left( \frac{a_1}{b_1} + \frac{a_2}{b_2} \right) + \frac{a_3}{b_3} = \frac{a_1b_2 + a_2b_1}{b_1b_2} + \frac{a_3}{b_3} = \frac{(a_1b_2b_3 + a_2b_1b_3) + a_3b_1b_2}{b_1b_2b_3}.$$ 

Since $a_1b_2b_3 + (a_2b_3b_1 + a_3b_2b_1) = (a_1b_2b_3 + a_2b_1b_3) + a_3b_1b_2$, we get $r_1 + (r_2 + r_3) = (r_1 + r_2) + r_3$. Also,

$$r_1(r_2r_3) = \frac{a_1}{b_1} \left( \frac{a_2}{b_2} \cdot \frac{a_3}{b_3} \right) = \frac{a_1(a_2a_3)}{b_1(b_2b_3)} = \frac{(a_1a_2)a_3}{(b_1b_2)b_3} = \frac{(a_1b_2b_3 + a_2b_1b_3) + a_3b_1b_2}{b_1b_2b_3}.$$ 

11. Assume that $x^2 = 5$ has a rational solution $a/b$. Then $a^2 = 5b^2$. The prime factor 5 will have an even exponent on the left and an odd exponent on the right, contradiction.

13. Let $S = \{am + bn : a, b \in \mathbb{Z}\} \cap \mathbb{N}$. Then $S$ is not empty (for example take $a = m, b = n$) and by the well ordering principle, it has a smallest element $d = a_0m + b_0n$ for some integers $a_0, b_0$. We want to show that $d = 1$. By the division algorithm, $m = dq + r$, where $0 \leq r < d$. Assuming $r \neq 0$, we see that $r = m - dq = m - q(a_0m + b_0n) \in S$ and $r < d$, contradiction. It remains that $d \mid m$. Similarly, $d \mid n$, so $d = 1$ since $m, n$ are relatively prime. Hence there are $a, b \in \mathbb{Z}$ with $1 = am + bn$.

14. Let $p \mid mn$. If $p \mid n$, we are done. If $p \nmid n$, then gcd($p, n$) = 1, so $1 = ap + bn$ for some integers $a, b$. Then $m = map + mbn$ and since $p \mid mn$, it follows that $p \mid m$. 

2. a) The set is not bounded above, so there is no least upper bound.
   b) The least upper bound is 1. Indeed, $1 - 1/n < 1$ for all $n$ and given $a < 1$, using the Archimedean property we can find $m$ such that $1/m < 1 - a$ i.e. $1 - 1/m > a$.
   c) $x^3 < 8$ is equivalent to $r < 2$ and the least upper bound is 2.
   d) We know that $-1 \leq \sin x \leq 1$ for all real $x$ and $\sin \frac{\pi}{2} = 1$, so the lub is 1.

4. $A = \left( \frac{-1 - \sqrt{5}}{2}, \frac{-1 + \sqrt{5}}{2} \right)$ and lub $A = \frac{-1 + \sqrt{5}}{2}$.

7. Given $x < y$ we can find $n \in \mathbb{N}$ such that $n > \frac{1}{y-x}$. Then $1 + nx < ny$. Let $m = \lfloor nx \rfloor \in \mathbb{Z}$. Then $m \leq nx < m + 1$ and $nx < m + 1 \leq 1 + nx < ny$. Dividing by $n$ we get $x < \frac{m+1}{n} < y$ and we can take $r = \frac{m+1}{n}$.

9. Given $r < s$ rational numbers, let $t \in \mathbb{Q}$ such that $\sqrt{2}r < t < \sqrt{2}s$. Then $r < \frac{t}{\sqrt{2}} < s$ and $t/\sqrt{2}$ is irrational.