HW3 solutions Math 310 Spring 2017

1.5

2. a) sup(−2, 8] = 8 = max(−2, 8], inf(−2, 8] = −2. No minimum.

b) sup\(\left\{ \frac{n + 2}{n^2 + 1} : n \in \mathbb{N} \right\}\) = \(\frac{3}{2}\) which is also a maximum; inf\(\left\{ \frac{n + 2}{n^2 + 1} : n \in \mathbb{N} \right\}\) = 0, no minimum.

c) Since \(-\sqrt{5} < \frac{n}{m} < \sqrt{5}\), sup\(\{n/m : n, m \in \mathbb{Z}, n^2 < 5m^2\}\) = \(\sqrt{5}\) and inf\(\{n/m : n, m \in \mathbb{Z}, n^2 < 5m^2\}\) = \(-\sqrt{5}\). The set has no maximum or minimum.

3. Given \(n \in \mathbb{N}\), sup \(A - 1/n\) is no longer an upper bound for \(A\), so there is \(a_n \in A\) with sup \(A - 1/n < a_n\). Obviously \(a_n \leq \sup A\).

4. Since sup \(A = \infty\), the set \(A\) is not bounded above. Given \(n \in \mathbb{N}\), since \(n\) is not an upper bound, there is \(a_n \in A\) with \(a_n > n\).

8. If \(A\) or \(B\) is not bounded above, then sup\(A \cup B\) = \(\infty = \max\{\sup A, \sup B\}\). Assume \(A, B\) bounded above. Since \(A, B \subset A \cup B\), we have sup \(A\), sup \(B \leq \sup(A \cup B)\), so max\(\{\sup A, \sup B\} \leq \sup(A \cup B)\). Let \(x \in A \cup B\). If \(x \in A\), then \(x \leq \sup A\). If \(x \in B\), then \(x \leq \sup B\). It follows that \(x \leq \max\{\sup A, \sup B\}\), so sup\(A \cup B\) \(\leq \max\{\sup A, \sup B\}\).

The proof for inf\(A \cup B\) = \(\min\{\inf A, \inf B\}\) is similar. We use \(A, B \subset A \cup B\) to get inf \(A\), inf \(B \geq \inf(A \cup B)\), hence min\(\{\inf A, \inf B\} \geq \inf(A \cup B)\). Given \(x \in A \cup B\) we have \(x \geq \inf A\) or \(x \geq \inf B\), so inf\(A \cup B\) \(\geq \min\{\inf A, \inf B\}\).

9. a) sup \(I x^2 = 1\), which is a maximum; inf \(I x^2 = 0\), which is a minimum.

b) sup \(I \frac{x + 1}{x - 1} = \infty\), not a maximum; inf \(I \frac{x + 1}{x - 1} = 3\), not a minimum.

c) sup \(I (2x - x^2) = 1\), not a maximum; inf \(I (2x - x^2) = 0\), a minimum.
2.1

3. a) $1, 3, 5, ..., 2n + 1, ...$
   
b) $1, -\frac{1}{2}, \frac{1}{4}, ..., (-1)^{n+1} \frac{1}{2n-1}, ...$
   
c) $1, \frac{1}{2}, \frac{1}{6}, ..., \frac{1}{n!}, ...$

5. $\lim_{n \to \infty} \frac{2n - 1}{3n + 1} = \frac{2}{3}$. Indeed, given $\varepsilon > 0$, let’s find $N$ such that $\left| \frac{2n - 1}{3n + 1} - \frac{2}{3} \right| < \varepsilon$ for $n > N$. But

$$\left| \frac{2n - 1}{3n + 1} - \frac{2}{3} \right| = \frac{5}{3(3n + 1)}$$

and the inequality becomes $\frac{5}{9n + 3} < \varepsilon$ or $n > \frac{5 - 3\varepsilon}{9\varepsilon}$. It suffices to take $N = \frac{5 - 3\varepsilon}{9\varepsilon}$.

8. We have

$$\sqrt{n+1} - \sqrt{n} = \frac{(\sqrt{n+1} + \sqrt{n})(\sqrt{n+1} - \sqrt{n})}{\sqrt{n+1} + \sqrt{n}} = \frac{n + 1 - n}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{n+1} + \sqrt{n}}.$$

It follows that $\lim(\sqrt{n+1} - \sqrt{n}) = 0$ because, given $\varepsilon > 0$ we can take $N = \frac{1}{4\varepsilon^2}$ and for $n > N$ we have

$$\sqrt{n+1} + \sqrt{n} > 2\sqrt{n} > 2\frac{1}{2\varepsilon} = \frac{1}{\varepsilon},$$

so $\frac{1}{\sqrt{n+1} + \sqrt{n}} < \varepsilon$.

11. For $k = 0$ it is clear. Assuming $k \neq 0$, since $a_n \to 0$, for all $\varepsilon > 0$ we can find $N$ such that

$$|a_n| < \frac{\varepsilon}{|k|} \text{ for } n > N.$$

It follows that $|ka_n| < \varepsilon$ for $n > N$, hence $ka_n \to 0$. 
5. We have
\[
\sqrt{n^2 + n - n} = \frac{(\sqrt{n^2 + n - n})(\sqrt{n^2 + n + n})}{\sqrt{n^2 + n + n}} = \frac{n}{\sqrt{n^2 + n + n}} = \frac{n}{n(\sqrt{1 + \frac{1}{n} + 1})} = \frac{1}{\sqrt{1 + \frac{1}{n} + 1}},
\]
so \(\lim(\sqrt{n^2 + n - n}) = \frac{1}{2}\) because
\[
\left|\frac{1}{\sqrt{1 + \frac{1}{n} + 1}} - \frac{1}{2}\right| = \frac{\frac{1}{2}(\sqrt{1 + \frac{1}{n} + 1} - 1)}{2}\left(\sqrt{1 + \frac{1}{n} + 1}\right)^2 < \frac{1}{8n}.
\]

Given \(\varepsilon > 0\) take \(N = \frac{1}{8\varepsilon}\) and for \(n > N\) we get
\[
\left|\sqrt{n^2 + n - n} - \frac{1}{2}\right| < \varepsilon.
\]

9. No, since for \(n = 6k\) we have \(\cos\frac{n\pi}{3} = 1\), for \(n = 6k \pm 1\) we have \(\cos\frac{n\pi}{3} = \frac{1}{2}\), for \(n = 6k + 3\) we have \(\cos\frac{n\pi}{3} = -1\) and for \(n = 6k \pm 4\) we have \(\cos\frac{n\pi}{3} = -\frac{1}{2}\).

10. If \(a_n = (-1)^n\), then \(\{a_n\}\) is divergent, but \(|a_n| = 1\) and \(|a_n|\) is convergent.

12. Let \(L = \lim a_n\). If \(L > c\), then for \(\varepsilon = L - c\) we can find \(N_1\) such that \(|a_n - L| < L - c\) for all \(n > N_1\). Hence \(c - L < a_n - L < L - c\), in particular \(a_n > c\) for all \(n > N_1\), which contradicts the fact that \(a_n \leq c\) for \(n\) large. It remains that \(L \leq c\). Similarly, if \(a_n \geq b\) for \(n\) large, assuming \(L < b\) we get a contradiction. It follows that \(L \geq b\).