2.5

5. a) No convergent subsequence since \( a_{2k} = 2^{2k} \to \infty \) and \( a_{2k+1} = -2^{2k+1} \to -\infty \).

b) We have \( a_{2k} = \frac{5 + 2k}{2 + 6k} \to \frac{1}{3}, \quad a_{2k+1} = \frac{5 - (2k + 1)}{2 + 3(2k + 1)} \to -\frac{1}{3} \). Two convergent subsequences.

c) We have \( a_{2k} = 2^{2k} \to \infty, \quad a_{2k+1} = -2^{-2k-1} \to 0 \). One convergent subsequence.

6. a) We have \( a_{2k} = 1 \) and \( a_{2k} \to 1 \).

b) We have \( a_{4k} = 0 \), so \( a_{4k} \to 0 \).

c) We have \( a_{2k} = \frac{2^k}{2k} - 1 = 0 \), so \( a_{2k} \to 0 \).

9. We may assume \( m > n \). Let \( m = n + k \). Then

\[
|a_m - a_n| \leq |a_{n+k} - a_{n+k-1}| + \cdots + |a_{n+1} - a_n| < 2^{-(n+k-1)} + \cdots + 2^{-n} = 2^{-n}(2^{-k+1} + \cdots + 1) = 2^{-n}\frac{1 - 2^{-k}}{1 - 1/2} < 2^{-n+1}.
\]

Given \( \varepsilon > 0 \), let \( N \) be such that \( 2^{-N+1} < \varepsilon \) i.e. \( N > 1 - \log_2 \varepsilon \).

Then for \( m, n > N \) we have \( |a_m - a_n| < \varepsilon \), so \( \{a_n\} \) is Cauchy.

10. No. Take \( a_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} \). Then \( a_{n+1} - a_n = \frac{1}{n+1} \) and given \( \varepsilon > 0 \), by taking \( N = 1 - 1/\varepsilon \) we get \( |a_{n+1} - a_n| < \varepsilon \) for \( n > N \).

Nevertheless, \( a_n \to \infty \) so it is not Cauchy.

11. Let \( m = n + k \). Then

\[
s_m - s_n = \sum_{k=n+1}^{n+k} \frac{1}{k2^k} < \sum_{k=n+1}^{n+k} \frac{1}{2^k} = \frac{1}{2^{n+1}} - \frac{1}{2^n} < \frac{1}{2^n}.
\]

Given \( \varepsilon > 0 \), choose \( N > \log_2(1/\varepsilon) \). Then for \( m, n > N \) we have \( |s_m - s_n| < \varepsilon \), so \( \{s_n\} \) is Cauchy and therefore convergent.
2.6

1. a) \(\liminf a_n = -1\), \(\limsup a_n = 1\).
   
b) \(\liminf a_n = \limsup a_n = 0\).
   
c) \(\liminf a_n = -\frac{\sqrt{3}}{2}\), \(\limsup a_n = \frac{\sqrt{3}}{2}\).

2. Given \(n \in \mathbb{N}\), there is a unique \(k = k_n\) such that \(2^k \leq n < 2^{k+1}\). It follows that \(\log_2 n - 1 < k_n \leq \log_2 n\), hence \(\frac{2}{3} < 2^{k_n} \leq n\) and \(0 \leq a_n < 1\). Since \(a_{2k} = 0\), we get \(\liminf a_n = 0\). Now \(a_{2k+1-1} = \frac{2^{k+1} - 1}{2^k} \rightarrow 1\), so \(\limsup a_n = 1\).

3. \(\liminf a_n = 0\) since there is a subsequence \(1, 1/2, 1/3, 1/4, ..., 1/m, ...\); \(\limsup a_n = 1\).

7. Let \(s_n = \sup\{a_k : k \geq n\}\), let \(t_n = \sup\{b_k : k \geq n\}\) and let \(u_n = \sup\{a_kb_k : k \geq n\}\). We will assume \(s_n, t_n, u_n \in \mathbb{R}\). For \(k \geq n\) we have \(a_k \leq s_n, b_k \leq t_n\), so \(a_kb_k \leq s_nt_n\).
   
   It follows that \(u_n \leq s_nt_n\), hence \(\lim u_n \leq (\lim s_n)(\lim t_n)\) and
   
   \[\limsup(a_nb_n) \leq (\limsup a_n)(\limsup b_n)\].

   Note that for \(a_n = n\) and \(b_n = 1/n\) we get \(1 \leq \infty \cdot 0\).

8. Let \(b_n \rightarrow b \neq 0\). If \(\{a_n\}\) is unbounded, then \(\limsup a_n = \infty\) and the equality follows. Assume \(\limsup a_n \in \mathbb{R}\). Since \(\limsup b_n = b\), from the previous problem we get \(\limsup(a_nb_n) \leq (\limsup a_n)b\).

   For the other inequality, let \(\{a_{n_k}\}\) be a subsequence such that \(a_{n_k} \rightarrow \limsup a_n\). Since \(b_{n_k} \rightarrow b\), we get \(a_{n_k}b_{n_k} \rightarrow (\limsup a_n)b\).

   Now \(\{a_{n_k}b_{n_k}\}\) is a subsequence of \(\{a_nb_n\}\), so \((\limsup a_n)b \leq \limsup(a_nb_n)\).

   Note that for \(a_n = n\) and \(b_n = 1/n\) we get \(1 = \infty \cdot 0\).