Infinite matrices and applications

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Matrices and Operators

A linear map (operator) $T : \mathbb{R}^n \to \mathbb{R}^n$ satisfies

$$T(x+y) = T(x) + T(y), \;\; T(cx) = cT(x)$$

for all $x, y \in \mathbb{R}^n$ and $c \in \mathbb{R}$.
A number λ is called an eigenvalue if there is a non zero vector v (called eigenvector) with $T(v) = \lambda v$. We associate to T an $n \times n$ matrix $A = (a_{ij}), i, j = 1, 2, ..., n$ with

$$T(e_j) = \sum_{i=1}^{n} a_{ij} e_i,$$

where $e_i = (0, ..., 0, 1, 0, ..., 0)$ with 1 on position i.

To find eigenvalues, we solve the equation $\det(A - \lambda I) = 0$. (We may get complex eigenvalues with complex eigenvectors).
We define the (operator) norm of T by

$$\|T\| = \sup\{\|T(x)\| : \|x\| = 1\}.$$

It measures "the maximum distorsion". Note that $|\lambda| \leq \|T\|$ for any eigenvalue λ.

Example: $T(x_1, x_2) = (x_1 + x_2, x_1)$ has matrix

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

with eigenvalues $(1 \pm \sqrt{5})/2$ and norm $(1 + \sqrt{5})/2$.

Recall that a vector space V has two operations, addition of vectors and multiplication with scalars, satisfying some properties. Linear maps $T : V \to V$, eigenvalues and eigenvectors are defined as above.

A basis $(e_j)_{j \in J}$ is a set of vectors such that

$$\sum_{j \in J} c_j e_j = 0 \Rightarrow c_j = 0$$

(linearly independent), and any other vector v is a linear combination

$$v = \sum_{j \in J} v_j e_j.$$

Here only a finite number of coefficients are non zero. The dimension of V is the number of elements of the set J.

Example: The space P_n of polynomials of degree $\leq n$ is a vector space of dimension $n+1$ with basis

$$1, x, x^2, ..., x^n.$$

The map

$$D : P_n \to P_n, (Dp)(x) = p'(x)$$

is a linear operator with matrix

$$\begin{pmatrix} 0 & 1 & 0 & ... & 0 \\ 0 & 0 & 2 & ... & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & ... & n \\ 0 & 0 & 0 & ... & 0 \end{pmatrix}.$$

The only eigenvalue of D is zero, with eigenvector 1.

To compute $\|D\|$, we use $\|D\|^2 = \|D^t D\|$ and the fact that, for a symmetric matrix, the norm is the absolute value of the largest eigenvalue.

$$D^t D = \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 4 & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \ldots & n^2 \end{pmatrix} \Rightarrow \quad \|D\| = n.$$

• For finite dimensional vector spaces each matrix gives a linear map, and every linear map is given by a matrix.
• For infinite dimensional vector spaces things are more complicated, and we need additional structure (norm, topology).

The infinite dimensional analogue of $\mathbb{R}^n$ is ℓ^2, where

$$\ell^2 = \{x = (x_1, x_2, x_3, ...) : \sum_{i=1}^{\infty} x_i^2 < \infty\},$$

with norm

$$\|x\| = (\sum_{i=1}^{\infty} x_i^2)^{1/2}.$$

This norm is associated to the inner product

$$x \cdot y = \sum_{i=1}^{\infty} x_i y_i,$$

and the space is complete, in the sense that every Cauchy sequence is convergent. ℓ^2 is a (real) Hilbert space. Not all infinite dimensional vector spaces are Hilbert spaces.

An infinite matrix is $A = (a_{ij})$ where $i, j \geq 1$. They appear in different contexts, including the quantization equation

$$AX - XA = I,$$

which has no solutions in finite matrices, or infinite linear systems of equations with infinitely unknowns, say

$$\sum_{j=1}^{\infty} a_{ij} x_j = y_i, \ i = 1, 2, 3, \ldots$$

Operations

$$A + B = (a_{ij} + b_{ij}), \ \ cA = (ca_{ij}), \ \ AB = (\sum_{k=1}^{\infty} a_{ik} b_{kj}).$$

We still talk about the identity matrix I, diagonal matrices, upper and lower triangular matrices, and the transpose $A^t = (a_{ji})$ of an infinite matrix. But

• For finite matrices, determinants play a fundamental role; no obvious analogue for infinite matrices.
• The product of infinite matrices may not be defined. We need convergence (analysis).
• Not every infinite matrix will have a finite norm.
• An infinite matrix may have a left inverse, but not a right inverse.

Examples

1) Let $V = \mathbb{R}[x]$ be the (infinite dimensional) vector space of polynomials with real coefficients, and let

$$S, D : V \to V, \;\; (Sp)(x) = xp(x), \;\; (Dp)(x) = p'(x).$$

With respect to the basis $1, x, x^2, x^3, ...,$ S has matrix

$$\begin{pmatrix} 0 & 0 & 0 & \ldots & 0 & 0 & \ldots \\ 1 & 0 & 0 & \ldots & 0 & 0 & \ldots \\ 0 & 1 & 0 & \ldots & 0 & 0 & \ldots \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \\ 0 & 0 & 0 & \ldots & 1 & 0 & \ldots \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \end{pmatrix}$$

and D has matrix

$$\begin{pmatrix} 0 & 1 & 0 & \ldots & 0 & \ldots \\ 0 & 0 & 2 & \ldots & 0 & \ldots \\ 0 & 0 & 0 & \ldots & 0 & \ldots \\ \vdots & \vdots & \vdots & & \vdots & \\ 0 & 0 & 0 & \ldots & n & \ldots \\ 0 & 0 & 0 & \ldots & 0 & \ldots \\ \vdots & \vdots & \vdots & & \vdots & \end{pmatrix}.$$

Note that S has no eigenvalue, and that $S^tS = I$, hence $\|S\| = 1$, but $SS^t \neq I$. The set of eigenvalues is replaced by the spectrum. The only eigenvalue of D is zero, with eigenvector 1. Moreover, $\|D\| = \infty$, if we use the ℓ^2-norm on V.

2) The matrix

$$\begin{pmatrix} 1 & 0 & 0 & \dots \\ \frac{1}{2} & \frac{1}{2} & 0 & \dots \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \dots \\ \vdots & \vdots & \vdots & \end{pmatrix}$$

with $a_{nk} = \frac{1}{n}$ for $1 \leq k \leq n$ and $a_{nk} = 0$ otherwise will transform a sequence $x_1, x_2, x_3, \dots$ into the sequence

$$x_1, \frac{x_1 + x_2}{2}, \frac{x_1 + x_2 + x_3}{3}, \dots$$

of arithmetic means.

3) The Hilbert matrix

$$H = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \cdots \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}$$

has entries $h_{ij} = \dfrac{1}{i+j-1}, i, j \geq 1$. It can be shown that H defines a bounded operator $H : \ell^2 \to \ell^2$ with norm π.

Reference: Man Duen Choi, Tricks or treats with the Hilbert matrix, Amer. Math. Monthly 90 (1983), no. 5, 301–312.

4) Let $\lambda_1, \lambda_2, \lambda_3, ...$ be a bounded sequence, and Λ the diagonal matrix with entries $\lambda_1, \lambda_2, \lambda_3, ...$ on the diagonal. Λ defines a bounded operator

$$T : \ell^2 \to \ell^2, T(x) = (\lambda_n x_n), \text{for } x = (x_n) \in \ell^2$$

with

$$\|T\| = \|\Lambda\| = \sup_{n \geq 1} |\lambda_n|.$$

All λ_n are eigenvalues of T with eigenvectors $e_n = (0, ..., 0, 1, 0, ...)$. If c denotes the (Banach) space of convergent sequences with norm $\|x\| = \sup_{n \geq 1} |c_n|$, then T defines also a bounded operator $T : c \to c$.

Toepliz and Stolz-Cesàro Theorems

Historical note. Otto Toeplitz (1881-1940) was a leading German mathematician, working on infinite linear and quadratic forms. He considered Banach spaces to be too abstract.

Otto Stolz (1842-1905) was an Austrian mathematician noted for his work on mathematical analysis and infinitesimals.

Ernesto Cesàro (1859-1906) was an Italian mathematician who worked in the field of differential geometry.

Theorem (Toeplitz). Let (a_{nk}) be a lower triangular ($a_{nk} = 0$ for $k > n$) infinite matrix such that

$$a_{nk} \geq 0, \quad \sum_{k=1}^{n} a_{nk} = 1 \text{ for all } n \geq 1, \text{and } \lim_{n\to\infty} a_{nk} = 0 \text{ for all } k \geq 1.$$

Then for any convergent sequence $(x_n)_{n\geq 1}$, the sequence $(y_n)_{n\geq 1}$, where

$$y_n = \sum_{k=1}^{n} a_{nk} x_k, \; n \geq 1$$

is convergent to the same limit.

If $\lim_{n\to\infty} x_n = \pm\infty$, then $\lim_{n\to\infty} y_n = \pm\infty$.

Proof (Case $x_n \to 0$). Let $\varepsilon > 0$ be fixed. There is $N \geq 1$ with $|x_n| \leq \varepsilon/2$ for all $n \geq N$. For $n > N$ we have

$$y_n = (a_{n1}x_1 + ... + a_{nN}x_N) + (a_{n,N+1}x_{N+1} + ... + a_{nn}x_n).$$

Since $\lim_{n\to\infty}(a_{n1}x_1 + ... + a_{nN}x_N) = 0$, there is $N' > N$ such that

$$|a_{n1}x_1 + ... + a_{nN}x_N| \leq \varepsilon/2 \text{ for all } n \geq N'.$$

For $n \geq N'$ we have

$$|y_n| \leq \frac{\varepsilon}{2} + \sum_{k=N+1}^{n} a_{nk}|x_k| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2}\sum_{k=1}^{n} a_{nk} = \varepsilon.$$

Hence $y_n \to 0$.

Corollary. Let $(x_n)_{n\geq 1}$ be a sequence with limit, and $(a_n)_{n\geq 1}$ a sequence of positive numbers with $\sum_{i=1}^{n} a_i \to \infty$. Then

$$\lim_{n\to\infty} \frac{a_1x_1 + ... + a_nx_n}{a_1 + ... + a_n} = \lim_{n\to\infty} x_n.$$

Proof. Apply Toeplitz Theorem for the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & ... \\ \frac{a_1}{a_1+a_2} & \frac{a_2}{a_1+a_2} & 0 & 0 & ... \\ \frac{a_1}{a_1+a_2+a_3} & \frac{a_2}{a_1+a_2+a_3} & \frac{a_3}{a_1+a_2+a_3} & 0 & ... \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}.$$

Theorem (Stolz-Cesàro). Let (a_n) and (b_n) two sequences of real numbers such that

1) $b_n > 0$, (b_n) is increasing and $\lim_{n\to\infty} b_n = \infty$;

2) $\lim_{n\to\infty} \frac{a_n - a_{n-1}}{b_n - b_{n-1}} = L \in \mathbb{R} \cup \{-\infty, \infty\}$.

Then $\lim_{n\to\infty} \frac{a_n}{b_n} = L$.

This is a discrete l'Hospital rule.

Proof. Consider the infinite matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & \dots \\ \frac{b_1}{b_2} & \frac{b_2-b_1}{b_2} & 0 & \dots & 0 & \dots \\ \vdots & \vdots & \vdots & & \vdots & \\ \frac{b_1}{b_n} & \frac{b_2-b_1}{b_n} & \frac{b_3-b_2}{b_n} & \dots & \frac{b_n-b_{n-1}}{b_n} & \dots \\ \vdots & \vdots & \vdots & & \vdots & \end{pmatrix}$$

and the sequence $x = (x_n)_{n\geq 1}$ with

$$x_1 = \frac{a_1}{b_1}, \;\; x_n = \frac{a_n - a_{n-1}}{b_n - b_{n-1}}, n \geq 2.$$

It is easy to check that $Ax = y$, where $y_n = \frac{a_n}{b_n}$ and that A satisfies the hypotheses of Toeplitz Theorem. We conclude that

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{a_n - a_{n-1}}{b_n - b_{n-1}} = L.$$

Applications.

1. Let $(x_n)_{n\geq 1}$ be a sequence with limit in $\mathbb{R}\cup\{-\infty,\infty\}$. Then

$$\lim_{n\to\infty}\frac{x_1+x_2+...+x_n}{n}=\lim_{n\to\infty}x_n.$$

Indeed, let $a_n=x_1+...+x_n$, $b_n=n$.

2. Let $x_n>0$. If $x_n\to x$, then $\sqrt[n]{x_1x_2\cdots x_n}\to x$. If $\dfrac{x_{n+1}}{x_n}\to x$, then $\sqrt[n]{x_n}\to x$.

3. $\displaystyle\lim_{n\to\infty}\frac{1^p+2^p+...+n^p}{n^{p+1}}=\frac{1}{p+1}$, p a positive integer.

Indeed, let $a_n=1^p+2^p+...+n^p$ and $b_n=n^{p+1}$. Then

$$\frac{a_{n+1}-a_n}{b_{n+1}-b_n}=\frac{(n+1)^p}{(n+1)^{p+1}-n^{p+1}}=\frac{n^p+pn^{p-1}+...+1}{(p+1)n^p+...+1}\to\frac{1}{p+1}.$$

4. $\lim\limits_{n\to\infty}(\frac{1^p+2^p+...+n^p}{n^p}-\frac{n}{p+1})=\frac{1}{2}$.

5. Let $0<x_0<1$ and $x_{n+1}=x_n-x_n^2$ for $n\geq 0$. Then $\lim\limits_{n\to\infty} nx_n=1$.
Indeed, $x_{n+1}<x_n$, and by induction $0<x_n<1$. Being bounded and monotonic, $x_n\to x$. Passing to the limit in the recurrence relation, $x=0$.

$$\lim_{n\to\infty} nx_n=\lim_{n\to\infty}\frac{n}{\frac{1}{x_n}}=\lim_{n\to\infty}\frac{n+1-n}{\frac{1}{x_{n+1}}-\frac{1}{x_n}}=\lim_{n\to\infty}(1-x_n)=1.$$

6. Fix $x_0\in(0,\pi)$ and let $x_{n+1}=\sin x_n,\ n\geq 0$. Then $\lim\limits_{n\to\infty}\sqrt{n}x_n=\sqrt{3}$. Hint: show first that $\lim\limits_{n\to\infty}\frac{1}{nx_n^2}=\frac{1}{3}$, using $a_n=\frac{1}{x_n^2}$ and $b_n=n$.

7. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that the sequence $a_n = \int_0^1 f(n+x)dx$ is convergent. Prove that the sequence $b_n = \int_0^1 f(nx)dx$ is also convergent. Hint: Show first that

$$b_n = \frac{\int_0^n f(x)dx}{n}.$$

8. Consider the curve $C : y = \sqrt{2x+1}, x > 0$ and the tangent line from $(0,k)$ to C which touches C at $(a_k, \sqrt{2a_k+1}), k \geq 2$. Show that

$$\lim_{n\to\infty} \frac{1}{n^3} \sum_{k=2}^{n} a_k = \frac{2}{3}.$$