14.3 Partial derivatives

\( f_x(a,b) = \lim_{h \to 0} \frac{f(a+h, b) - f(a, b)}{h} = \frac{\partial f}{\partial x}(a, b) \)

\( f_y(a, b) = \lim_{h \to 0} \frac{f(a, b+h) - f(a, b)}{h} = \frac{\partial f}{\partial y}(a, b) \)

\( f_x(a, b) \) represents the slope of the tangent line to the curve obtained by cutting with the plane \( y = b \)

Examples

1) \( f(x, y) = x^2 + 3xy + y - 1 \)  \( (a, b) = (2, 3) \)

\( f_x = 2x + 3y \quad f_y = 3x + 1 \)

\( f_x(2, 3) = 2(2) + 3(3) = 13 \quad f_y(2, 3) = 3(2) + 1 = 7 \)

2) \( g(x, y) = x \cos(xy) \)

\( g_x = \cos(xy) + x(-\sin(xy)) \quad g_y = \cos(xy) - xy \sin(xy) \)

3) \( h(x, y) = e^{xy} \ln(x-y) \)

\( h_x = y e^{xy} \ln(x-y) + e^{xy} \frac{1}{x-y} \)

\( h_y = xe^{xy} \ln(x-y) + e^{xy} \frac{-1}{x-y} \)
3) \[ h(x,y) = e^{xy} \ln(x-y) \]

\[ h_x = \frac{\partial h}{\partial x} = e^{xy} \cdot y \cdot \ln(x-y) + e^{xy} \cdot \frac{1}{x-y} \]

\[ h_y = \frac{\partial h}{\partial y} = e^{xy} \cdot x \ln(x-y) + e^{xy} \cdot \frac{1}{x-y} \]  

Partial differential equations

Laplace eq. \[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \]

\[ \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial x} \right) \]

\[ u_{xx} = \left( u_x \right)_x \]

Wave eq. \[ \frac{\partial^2 u}{\partial t^2} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \]

\[ u(x,y) = x^2 - y^2 \] is a solution to Laplace eq.

\[ u_x = 2x \quad u_{xx} = 2 \quad u_y = -2y \quad u_{yy} = -2 \quad 2 + (-2) = 0 \]

\[ u(t,x) = \sin((x-\alpha t) + \ln(x+\alpha t)) \] is a solution to the wave eq.

\[ \frac{\partial u}{\partial t} = \cos(x-\alpha t) \cdot (-\alpha) + \frac{1}{x+\alpha t} \cdot \alpha \]

\[ \frac{\partial^2 u}{\partial t^2} = -\sin(x-\alpha t)(-\alpha)(-\alpha) - \frac{a \cdot a}{(x+\alpha t)^2} = -a^2 \sin(x-\alpha t) - \frac{a^2}{(x+\alpha t)^2} \]

\[ \frac{\partial u}{\partial x} = \cos(x-\alpha t) + \frac{1}{x+\alpha t} \]

\[ \frac{\partial^2 u}{\partial x^2} = -\sin(x-\alpha t) - \frac{1}{(x+\alpha t)^2} \]

\[ \frac{\partial^2 u}{\partial t^2} = a^2 \left( \sin(x-\alpha t) - \frac{1}{(x+\alpha t)^2} \right) = a^2 \frac{\partial^2 u}{\partial x^2} \]

Higher partial derivatives

let \[ z = f(x,y) \]

\[ f_{xx} = (f_x)_x \quad f_{xy} = (f_x)_y \]

\[ f_{yx} = (f_y)_x \quad f_{yy} = (f_y)_y \]

Theorem. If \( f_{xy}, f_{yx} \) are continuous, then \( f_{xy} = f_{yx} \).
Example: \( y(x, y) = x \cos(xy) \)

\[
g_x = \cos(xy) - xy \sin(xy)
\]

\[
g_{xy} = (g_x)_y = -x \sin(xy) - x \sin(xy) - xy \cos(xy)
\]

\[= -2x \sin(xy) - xy \cos(xy)\]

\[
g_y = -x^2 \sin(xy)
\]

\[
g_{yx} = (g_y)_x = -2x \sin(xy) - x^2 \cos(xy)
\]

\[
g_{xyx} = g_{xyx} = (g_{xy})_x = -2 \sin(xy) - 2xy \cos(xy) - 2xy \cos(xy)
\]

\[+ x^2 \sin(xy) = -2 \sin(xy) - 4xy \cos(xy) + x^2 \sin(xy)\]

**Implicit Differentiation**

Given an equation involving \( x, y, z \) which gives \( z \) as a function of \( x, y \) implicitly, we compute \( \frac{\partial z}{\partial x} \), \( \frac{\partial z}{\partial y} \) by implicit differentiation:

\[
xz = \ln(yz) + x - y \quad \text{gives} \quad z = z(x, y)
\]

To find \( \frac{\partial z}{\partial x} \), differentiate both sides with respect to \( x \):

\[
1 + x \cdot \frac{\partial z}{\partial x} = \frac{1}{yz} \cdot y \frac{\partial z}{\partial x} + 1
\]

Solve for \( \frac{\partial z}{\partial x} \):

\[
x \frac{\partial z}{\partial x} - \frac{1}{z} \frac{\partial z}{\partial x} = 1 - z \quad \text{or} \quad \frac{\partial z}{\partial x} \left( x - \frac{1}{z} \right) = 1 - z\]

\[
\frac{\partial z}{\partial x} = \frac{1 - z}{x - \frac{1}{z}}.
\]

Let's find \( \frac{\partial z}{\partial y} \):

\[
x \frac{\partial z}{\partial y} = \frac{1}{yz} \left( z + y \frac{\partial z}{\partial y} \right) - 1
\]

\[
x \frac{\partial z}{\partial y} = \frac{1 - z}{y} - 1 \quad \Rightarrow \quad \frac{\partial z}{\partial y} = \frac{\frac{1}{y} - 1}{x - \frac{1}{z}}.
\]