14.8 Method of Lagrange multipliers

Objective \( f(x_1 y), \ f(x_1 y_1 \gamma) \)

Constraint \( g(x_1 y), \ g(x, y_1 \gamma) \).

Maximize or minimize \( f \) subject to \( g = k \).

We solve the system
\[
\begin{align*}
\nabla f &= \lambda \nabla g \\
1 &= k
\end{align*}
\]

Example. Objective \( f(x_1 y) = x^2 + 2y^2 \), constraint \( g(x_1 y) = x + y = 1 \)

\( \nabla f = \langle 2x, 4y \rangle \quad \nabla g = \langle 2, 1 \rangle \)

\[
\begin{align*}
2x &= \lambda - 2x \quad \Rightarrow \quad x = 0 \quad \text{or} \quad \lambda = 1 \\
4y &= \lambda - 2y \quad \Rightarrow \quad y = 0 \quad \text{or} \quad \lambda = 2 \\
x^2 + y^2 &= 1 \quad \text{Case 1:} \quad x = 0 \quad y^2 = 1 \quad \Rightarrow \quad y = \pm 1 \\
&\quad \text{Case 2:} \quad y = 0 \quad x^2 = 1 \quad \Rightarrow \quad x = \pm 1
\end{align*}
\]

Solutions \((0, \pm 1), (\pm 1, 0)\).

\( f(0, \pm 1) = 2 \quad f(\pm 1, 0) = 1 \quad f_{\text{max}} = 2 \quad f_{\text{min}} = 1 \)

41. Find points on \( z^2 = x^2 + y^2 \) closest to \((4, 2, 0)\).

\[
\text{dist} = \sqrt{(x-4)^2 + (y-2)^2 + z^2} \quad \text{Objective} \quad f(x, y, z) = (x-4)^2 + (y-2)^2 + z^2
\]

Constraint \( g(x_1 y_1 z) = x^2 + y^2 - z^2 = 0 \)

\[
\begin{align*}
2(x-4) &= \lambda \cdot 2x \\
2(y-2) &= \lambda \cdot 2y \\
2z &= \lambda (-2z) \quad \Rightarrow \quad z = 0 \quad \text{or} \quad \lambda = -1 \\
x^2 + y^2 - z^2 &= 0
\end{align*}
\]

If \( z = 0 \), then \( x^2 + y^2 = 0 \), hence \( x = 0, y = 0 \)

\( x = 0 \) does not satisfy the first eq.
If \( x = -1 \), then \( 2x - 8 = -2x \Rightarrow x = 2 \)
and \( 2y - 4 = -2y \Rightarrow y = 1 \).

\[
2^2 + 1^2 - z^2 = 0 \Rightarrow z^2 = 5 \Rightarrow z = \pm \sqrt{5}
\]

Solutions \( (2, 1, \pm \sqrt{5}) \).

\[
\mathbf{f}(2, 1, \pm \sqrt{5}) = (2-4)^2 + (1-2)^2 + (\pm \sqrt{5})^2 = 10 = \mathbf{f}_{\text{min}}.
\]

\[
\text{dist} = \sqrt{10}
\]

9. Rectangular box with one vertex at \((0, 0, 0)\), faces in the coord. planes in the first octant and one vertex on the ellipsoid \( x^2 + 2y^2 + 3z^2 = 6 \).

Find the dimensions which give a max. volume

\[
\mathbf{f}(x, y, z) = xyz \quad \mathbf{g}(x, y, z) = x^2 \quad 2y^2 + 3z^2 = 6
\]

\[
\begin{align*}
yz &= \lambda \cdot 2x \\
xz &= \lambda \cdot 4y \\
xy &= \lambda \cdot 6z \\
x^2 + 2y^2 + 3z^2 &= 6
\end{align*}
\]

\[
\lambda = \frac{yz}{2x} = \frac{xz}{4y} = \frac{xy}{6z}
\]

\[
4y^2 z = 2x^2 z \quad 6x^2 z = 4xyz \quad x^2 = 2y^2 \\
y^2 = \frac{3}{2} z^2 
\Rightarrow \quad x^2 = 2 \cdot \frac{3}{2} z^2 = 3z^2
\]

\[
3z^2 + 3z^2 + 3z^2 = 6 
\Rightarrow \quad z^2 = \frac{2}{3} \Rightarrow z = \pm \sqrt[3]{\frac{2}{3}} \quad z = \sqrt[3]{\frac{2}{3}}
\]

\[
x = \sqrt{2} \quad y = 1
\]

\[
\mathbf{f}_{\text{max}} = \sqrt{2} \cdot 1 \cdot \frac{\sqrt{2}}{\sqrt[3]{3}} = \frac{2}{\sqrt[3]{3}}.
\]
Objective \( f(x, y, z) = xyz \)

Constraint \( g(x, y, z) = xy + 2xz + 2yz = 12 \)

\[
\begin{align*}
\frac{yz}{y + 2z} &= \frac{xz}{x + 2z} = \frac{xy}{2x + 2y} \\
xy &= \lambda (x + 2z) = \lambda (2x + 2y) = 12 \\
xz &= \lambda (x + 2z) = \lambda (2x + 2y) = 12 \\
yz &= \lambda (y + 2z) = \lambda (2x + 2y) = 12
\end{align*}
\]

\( \frac{yz}{y + 2z} = \frac{xz}{x + 2z} = \frac{xy}{2x + 2y} \)

\( \frac{yz}{y + 2z} = \frac{xz}{x + 2z} \)

\( xz (2x + 2y) = xy (x + 2z) \)

\( 2yz = 2xz \Rightarrow x = y \)

\( 2xz = xy \Rightarrow z = \frac{y}{2} \)

\( y^2 + 2y \cdot \frac{y}{2} + 2y \cdot \frac{y}{2} = 12 \)

\( 3y^2 = 12 \Rightarrow y = 2, x = 2, z = 1 \)

\( V_{\text{max}} = 2 \cdot 2 \cdot 1 = 4 \text{ m}^3 \)
\[ f(x,y) = x^2 + y^2 + 4x - 4y \quad x^2 + y^2 \leq 9. \]

\[ f_{\text{min}} = ? \quad f_{\text{max}} = ? \]

\[
\begin{align*}
 f_x &= 2x + 4 = 0 \quad \Rightarrow \quad x = -2 \\
 f_y &= 2y - 4 = 0 \quad \Rightarrow \quad y = 2
\end{align*}
\]

Critical pt \((-2,2)\) inside

On the boundary \(x^2 + y^2 = 9\) we use Lagrange multipliers:

\[
\begin{cases}
 2x + 4 = \lambda \cdot 2x \\
 2y - 4 = \lambda \cdot 2y \\
 x^2 + y^2 = 9
\end{cases}
\]

Observation: \(x \neq 0\), \(y \neq 0\), so we can solve for \(\lambda\) in both equations:

\[
\lambda = \frac{2x + 4}{2x} = \frac{2y - 4}{2y}
\]

\[
\frac{2x + 4}{2x} = \frac{2y - 4}{2y} \quad 4xy + 8y = 4xy - 8x \quad \Rightarrow \quad y = -x
\]

\[
x^2 + (-x)^2 = 9 \quad 2x^2 = 9 \quad x^2 = \frac{9}{2} \quad x = \pm \frac{3}{\sqrt{2}}
\]

Solutions: \(\left(\frac{3}{\sqrt{2}}, \frac{-3}{\sqrt{2}}\right), \left(-\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right)\)

\[ f\left(\frac{3}{\sqrt{2}}, \frac{-3}{\sqrt{2}}\right) = 9 + \frac{24}{\sqrt{2}} \quad \text{max at } \left(\frac{3}{\sqrt{2}}, \frac{-3}{\sqrt{2}}\right) \]

\[ f\left(-\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right) = 9 - \frac{24}{\sqrt{2}} \]

\[ f_{\text{min}} = -8 \quad \text{at } (-2,2). \]