1. Let $f : [-2,3] \to \mathbb{R}, f(x) = x^3 - 3x$. Find the intervals on which $f$ is increasing or decreasing. Find its maximum and minimum. Use the Mean Value Theorem to show that there is $c \in (-1,1)$ such that $f'(c) = -2$.

Since $f'(x) = 3x^2 - 3$, the critical points are $\pm 1$ and the endpoints. $f$ is increasing on $[-2,-1] \cup [1,3]$, where $f'(x) \geq 0$, and decreasing on $[-1,1]$, where $f'(x) \leq 0$. We have

$$f(-2) = -2 = f(1), \quad f(-1) = 2, \quad f(3) = 18,$$

so $f_{\text{max}} = 18, f_{\text{min}} = -2$. Applying MVT on $[-1,1]$ we get $c \in (-1,1)$ such that

$$f'(c) = \frac{f(1) - f(-1)}{1 - (-1)} = \frac{-2 - 2}{2} = -2.$$

2. Consider $g(x) = \begin{cases} x \cos \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$ Prove that $g$ is continuous. Is $g$ differentiable at $a = 0$?

Obviously, $g$ is continuous for $x \neq 0$. Since $|x \cos \frac{1}{x}| \leq |x|$, we get $\lim_{x \to 0} g(x) = 0 = g(0)$, so $g$ is also continuous at 0. But $g$ is not differentiable at 0 since

$$\lim_{x \to 0} \frac{g(x) - g(0)}{x - 0} = \lim_{x \to 0} \cos \frac{1}{x}$$

does not exist. Indeed, for $x_n = \frac{1}{2n\pi}$ we have $\cos \frac{1}{x_n} = 1$ and for $y_n = \frac{1}{(2n+1)\pi} \to 0$ we have $\cos \frac{1}{y_n} = -1$.

3. Evaluate the limits 

a. $\lim_{x \to 0} \frac{1 - \cos x}{2x^2}$; 

b. $\lim_{x \to 1^-} x^{\frac{1}{x}}$.

a. We are in a $\frac{0}{0}$ case. Using l’Hôpital’s rule, $\lim_{x \to 0} \frac{1 - \cos x}{2x^2} = \lim_{x \to 0} \frac{\sin x}{4x} = \frac{1}{4}$ since

$$\lim_{x \to 0} \frac{\sin x}{x} = 1.$$

b. We are in a $1^\infty$ case. Since

$$x^{\frac{1}{x-1}} = e^{\ln x^{\frac{1}{x-1}}} = e^{\frac{\ln x}{x-1}}$$

$$= e^{\ln x \frac{1}{x-1}} = e^{\frac{\ln x}{x-1}},$$
and \( \lim_{x \to 1^-} \frac{\ln x}{1 - x} = \lim_{x \to 1^-} \frac{1}{x} = -1 \), we get \( \lim_{x \to 1^-} x^{1/x} = e^{-1} = 1/e \).

4. Let \( h : [\pi/2, \pi) \to (-\infty, 0], \) \( h(x) = \cot x = \frac{\cos x}{\sin x} \). Prove that \( h \) is one-to-one and onto. Use the identity \( 1 + \cot^2 x = \csc^2 x \) to find a formula for \( (h^{-1})'(y) \).

Since \( h'(x) = -\csc^2 x < 0 \), the function is strictly decreasing, hence one-to-one. We have \( h(\pi/2) = 0 \) and \( \lim_{x \to \pi^-} h(x) = -\infty \), so \( h \) is onto. Now

\[
(h^{-1})'(y) = \frac{1}{h'(h^{-1}(y))} = \frac{1}{-\csc^2(\cot^{-1}(y))} = -\frac{1}{1 + \cot^2(\cot^{-1}(y))} = -\frac{1}{1 + y^2}.
\]

5. Show that the sequence of functions \( f_n(x) = x^{1/n} \) does not converge uniformly on \([0, 1]\).

We have \( \lim_{n \to \infty} f_n(x) = f(x) = \begin{cases} 1 & \text{if } 0 < x \leq 1 \\ 0 & \text{if } x = 0. \end{cases} \)

Since \( f_n \) are continuous on \([0, 1]\) but the pointwise limit function \( f \) is not, the convergence is not uniform.