1. Suppose that $A$ is a finite set, and $f : A \rightarrow A$ is one-to-one. Prove that $f$ is onto.

Indeed, if $A$ has $n$ elements, then $f(A) \subset A$ also has $n$ elements, hence $f(A) = A$.

2. Let $f(x) = \begin{cases} -1 & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}$. Find $f([-2, 2])$, $f^{-1}([-2, 2])$, $f^{-1}(f((\infty, 0)))$.

$f([-2, 2]) = \{-1\} \cup [0, 2]$, $f^{-1}([-2, 2]) = (-\infty, 2]$ and $f^{-1}(f((\infty, 0))) = (-\infty, 0]$.

3. Let $f : X \rightarrow Y$ be a function which is one-to-one and onto. Prove that there is a function $g : Y \rightarrow X$ such that $f \circ g = id_{Y}$ and $g \circ f = id_{X}$.

For $y \in Y$, define $g(y) = x$ where $x \in X$ is such that $f(x) = y$. Then $x$ is unique, so $g$ is well defined and $(f \circ g)(y) = y$, $(g \circ f)(x) = x$.

4. Let $S$ be a bounded nonempty subset of $\mathbb{R}$. For $a \in \mathbb{R}$ define $a + S = \{a + s : s \in S\}$. Show that $\sup(a + S) = a + \sup S$ and $\inf(a + S) = a + \inf S$.

Given $s \in S$ we have $s \leq \sup S$, so $a + s \leq a + \sup S$. We get $\sup(a + S) \leq a + \sup S$. From $a + s \leq \sup(a + S)$ we get $s \leq \sup(a + S) - a$, hence $\sup S \leq \sup(a + S) - a$ which gives the other inequality. The equality about inf is proved similarly.

5. Let $\{x_n\}$ be such that $x_1 = 0$ and $x_{n+1} = \frac{1}{1+x_n}$. Prove by induction that $x_{n+2}$ is between $x_n$ and $x_{n+1}$ for all $n$. Is the sequence convergent?

Assuming $x_k \leq x_{k+2} \leq x_{k+1}$, by adding 1 and taking reciprocals, we get $x_{k+1} \geq x_{k+3} \geq x_{k+2}$.

Similarly, $x_k \geq x_{k+2} \geq x_{k+1}$ gives $x_{k+1} \leq x_{k+3} \leq x_{k+2}$. $\lim x_n = \frac{-1 + \sqrt{5}}{2}$.

6. Prove by induction that the sum of the cubes of three consecutive natural numbers is divisible by 9.

Assuming $(k - 1)^3 + k^3 + (k + 1)^3 = 3k^3 + 6k$ divisible by 9, we get $k^3 + (k + 1)^3 + (k + 2)^3 = 3k^3 + 9k^2 + 15k + 9 = (3k^3 + 6k) + 9(k^2 + k + 1)$ is divisible by 9.
7. Prove that $\sqrt{2} + \sqrt{3}$ is irrational.

Show first that $\sqrt{6}$ is irrational. If $\sqrt{2} + \sqrt{3} = a/b$, squaring both sides we get

$$5 + 2\sqrt{6} = \frac{a^2}{b^2},$$

so $\sqrt{6}$ is rational, contradiction.

8. Let $f : (0, 2] \to \mathbb{R}, f(x) = \frac{x}{2} + \frac{1}{x}$. Find its supremum and infimum.

$$\inf_{x \in [0,2]} f = \sqrt{2}, \sup_{x \in [0,2]} f = \infty.$$

9. Let $x_1 = 4$ and $x_{n+1} = 3 - (1/x_n)$ for $n \geq 1$. Show that $\{x_n\}_{n \in \mathbb{N}}$ converges and find its limit.

Assuming $x_{k+1} \leq x_k$, we get $1/x_{k+1} \geq 1/x_k$, so $3 - 1/x_{k+1} \leq 3 - 1/x_k$ and by induction, the sequence is decreasing and $x_n \geq 2$. $\lim x_n = \frac{3 + \sqrt{5}}{2}$.

10. Prove that a sequence $\{a_n\}$ such that $|a_{n+1} - a_n| < 2^{-n}$ for all $n$ is Cauchy.

We may assume $m > n$. Let $m = n + k$. Then

$$|a_m - a_n| \leq |a_{n+k} - a_{n+k-1}| + \cdots + |a_{n+1} - a_n| < 2^{-(n+k-1)} + \cdots + 2^{-n} = 2^{-n}(2^{-k+1} + \cdots + 1) = 2^{-n} \frac{2^k - 1}{2^k - 2} < 2^{-n+1}.$$

Given $\varepsilon > 0$, let $N$ be such that $2^{-N+1} < \varepsilon$ i.e. $N > 1 - \log_2 \varepsilon$.

11. Find $\lim \inf x_n$ and $\lim \sup x_n$ if $x_n = (-1)^n(2 + \frac{1}{n}) \sin \frac{n\pi}{3}$.

$\lim \inf x_n = -\sqrt{3}$ and $\lim \sup x_n = \sqrt{3}$.

12. Compute the limits

$$\lim_{n \to \infty} \frac{\sin n^2}{\sqrt{n}}, \quad \lim_{n \to \infty} \frac{2^n}{n!}, \quad \lim_{n \to \infty} \left(\sqrt{n^2 + n} - n\right).$$

a) Since $-1 \leq \sin n^2 \leq 1$, the limit is 0 by the Squeeze Principle.

b) Let $a_n = \frac{2^n}{n!}$. Since $a_{n+1} = \frac{2}{n+1}a_n$, it follows that $\{a_n\}$ is decreasing and $\lim a_n = 0$.

c) Multiplying with the conjugate, $\lim_{n \to \infty} \left(\sqrt{n^2 + n} - n\right) = \frac{1}{2}$. 

13. Give examples of sequences \( \{x_n\}_{n \in \mathbb{N}} \) and \( \{y_n\}_{n \in \mathbb{N}} \) such that

a) \( \{x_n\}_{n \in \mathbb{N}} \) is bounded and \( \{y_n\}_{n \in \mathbb{N}} \) converges, but \( \{x_n y_n\}_{n \in \mathbb{N}} \) does not converge.

b) \( x_n \to \infty \) and \( y_n \to 0 \) but \( \{x_n y_n\}_{n \in \mathbb{N}} \) does not converge.

a) Let \( x_n = (-1)^n \), \( y_n = 1 \).

b) Let \( x_n = n \), \( y_n = \frac{(-1)^n}{n} \).

14. If \( \limsup a_n \) and \( \limsup b_n \) are finite, prove that \( \limsup(a_n + b_n) \leq \limsup a_n + \limsup b_n \) and show that the inequality may be strict.

Use the definition of \( \limsup \). Take \( a_n = (-1)^n \), \( b_n = (-1)^n + 1 \).

15. Let \( x_1 > 0 \) and let \( x_{n+1} = 2/(3 + x_n) \) for \( n \geq 1 \). Show that \( \{x_n\}_{n \in \mathbb{N}} \) is Cauchy, and find its limit.

Note that \( x_n > 0 \). Since

\[
x_{n+2} - x_{n+1} = \frac{2}{3 + x_{n+1}} - \frac{2}{3 + x_n} = \frac{2(x_n - x_{n+1})}{(3 + x_{n+1})(3 + x_n)}
\]

we get

\[
|x_{n+2} - x_{n+1}| < \frac{2}{9} |x_{n+1} - x_n| < \cdots < \frac{2^n}{9^n} |x_2 - x_1|
\]

and

\[
|x_{m+k+1} - x_{n+1}| \leq |x_{n+k+1} - x_{n+k}| + \cdots + |x_{n+2} - x_{n+1}| < |x_2 - x_1| \frac{2^n}{9^n} \left( \frac{2^{k-1}}{9^{k-1}} + \cdots + 1 \right) < |x_2 - x_1| \frac{2^n}{9^n} \frac{1}{1 - \frac{2}{9}}.
\]

Given \( \varepsilon > 0 \), choose \( N \) such that \( N > \log_2 \frac{7\varepsilon}{9|x_2 - x_1|} \).