4.18 Find the maximum likelihood estimates of the $2\times1$ mean vector $\mu$ and the $2\times2$ covariance matrix $\Sigma$ based on the random sample

$$X = \begin{bmatrix} 3 & 6 \\ 4 & 4 \\ 5 & 7 \\ 4 & 7 \end{bmatrix}$$

from a bivariate normal population.

**Sln:** Since the random samples $X_1, X_2, X_3$, and $X_4$ are from normal population, the MLEs of $\mu$ and $\Sigma$ are $\bar{X}$ and $\frac{1}{n} \sum_{j=1}^{n} (\bar{X}_j - \bar{X})(\bar{X}_j - \bar{X})^T$. Therefore,

$$\hat{\mu} = \frac{1}{n} X^T, \quad \hat{\Sigma} = \frac{1}{n} \sum_{j=1}^{n} (X_j - \bar{X})(X_j - \bar{X})^T$$

$$= \frac{1}{4} \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} 2 & 1 \\ 1 & 6 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} \end{bmatrix}$$

4.19 Let $X_1, X_2, \ldots, X_{20}$ be a random sample of size $n = 20$ from an $N_6(\mu, \Sigma)$ population. Specify each of the following completely.

(a) the distribution of $(X_i - \mu)^T \Sigma^{-1} (X_i - \mu)$

(b) the distributions of $\bar{X}$ and $\sqrt{n}(\bar{X} - \mu)$

(c) the distribution of $(n-1)S$

**Sln:** (a) $(X_i - \mu)^T \Sigma^{-1} (X_i - \mu)$ is distributed as $\chi^2_6$.

(b) $\bar{X}$ is distributed as $N_6(\mu, \frac{1}{20} \Sigma)$ and $\sqrt{n}(\bar{X} - \mu)$ is distributed as $N_6(0, \Sigma)$. 

1
(c) \((n-1)S\) is distributed as Wishart distribution \(\sum_{i=1}^{20-1} Z_i Z_i^T\), where \(Z_i \sim N_6(0, \Sigma)\).

4.21 Let \(X_1, X_2, \ldots, X_{60}\) be a random sample of size 60 from a four-variate normal distribution having mean \(\mu\) and covariance \(\Sigma\). Specify each of the following completely.

(a) The distribution of \(\bar{X}\)

(b) The distribution of \(\left( X_i - \mu \right)^T \Sigma^{-1} \left( X_i - \mu \right)\)

(c) The distribution of \(n \left( \bar{X} - \mu \right)^T \Sigma^{-1} \left( \bar{X} - \mu \right)\)

(d) The approximate distribution of \(n \left( \bar{X} - \mu \right)^T S^{-1} \left( \bar{X} - \mu \right)\)

Sln: (a) \(\bar{X}\) is distributed as \(N_4(\mu, \frac{1}{60} \Sigma)\)

(b) \(\left( X_i - \mu \right)^T \Sigma^{-1} \left( X_i - \mu \right)\) is distributed as \(\chi^2_4\).

(c) \(n \left( \bar{X} - \mu \right)^T \Sigma^{-1} \left( \bar{X} - \mu \right)\) is distributed as \(\chi^2_4\), because \(\bar{X}\) is distributed as \(N_4(\mu, \frac{1}{60} \Sigma)\)

(d) Since \(60 \gg 4\), \(n \left( \bar{X} - \mu \right)^T S^{-1} \left( \bar{X} - \mu \right)\) can be approximated as \(\chi^2_4\).

4.22 Let \(X_1, X_2, \ldots, X_{75}\) be a random sample from population distribution with mean \(\mu\) and covariance \(\Sigma\). What is the approximate distribution of each of the following.

(a) \(\bar{X}\)

(b) \(n \left( \bar{X} - \mu \right)^T S^{-1} \left( \bar{X} - \mu \right)\)

Sln: (a) \(\bar{X}\) can be approximated by \(N_p(\mu, \frac{1}{75} \Sigma)\)

(b) \(n \left( \bar{X} - \mu \right)^T S^{-1} \left( \bar{X} - \mu \right)\) can be approximated by \(\chi^2_p\)