Below we assume that $Z_t \sim WN(0, \sigma^2)$.

3.1 Consider MA(1) process $X_t = a Z_t + b Z_{t-1}$. Find the white noise $W_t$ such that the process $X_t$ is presented as $X_t = W_t + \theta W_{t-1}$ with $W_t \sim WN(0, \gamma^2)$.

**Solution:** From class we know the general solution for the acvf of an MA(1) process. Thus we can express the acvf for either of the MA(1) representations of $X_t$ listed in the problem statement.

$$\gamma_x(h) = \begin{cases} (a^2 + b^2)\sigma^2, & h = 0 \\ ab\sigma^2, & |h| = 1 \\ 0, & |h| \geq 2 \end{cases}$$

$$\gamma_x(h) = \begin{cases} (1 + \theta^2)\gamma^2, & h = 0 \\ \theta\gamma^2, & |h| = 1 \\ 0, & |h| \geq 2 \end{cases}$$

We can solve the resulting system of equations as follows

$$ab\sigma^2 = \theta\gamma^2 \implies \gamma^2 = \frac{ab}{\theta} \sigma^2$$

$$(a^2 + b^2)\sigma^2 = (1 + \theta^2)\gamma^2$$

$$= (1 + \theta^2) \frac{ab}{\theta} \sigma^2$$

$$ab\theta^2 - (a^2 + b^2)\theta + ab = 0$$

$$\theta = \frac{a}{b}, \frac{b}{a}$$

Thus the white noise $W_t$ can have either of the following white noise distributions: $WN(0, a^2\sigma^2)$ or $WN(0, b^2\sigma^2)$.

3.2 Find acvf and acf for MA(1), MA(2), MA(3).

**Solution:**

(1) MA(1): $X_t = Z_t + \theta_1 Z_{t-1}$
\[ \gamma_x(h) = E\{[X_t - E(X_t)][X_{t+h} - E(X_{t+h})]\} = E(X_t X_{t+h}) \]
\[ = E\{[Z_t + \theta_1 Z_{t-1}][Z_{t+h} + \theta_1 Z_{t+h-1}]\} = E(Z_t Z_{t+h}) + \theta_1 E(Z_t Z_{t+h-1}) + \theta_1^2 E(Z_{t-1} Z_{t+h}) \]
\[ = \begin{cases} 
(1 + \theta_1^2)\sigma^2, & h = 0 \\
\theta_1\sigma^2, & |h| = 1 \\
0, & |h| \geq 2 
\end{cases} \]
\[ \rho_x(h) = \frac{\gamma_x(h)}{\gamma_x(0)} = \begin{cases} 
1, & h = 0 \\
\frac{\theta_1}{1 + \theta_1^2}, & |h| = 1 \\
0, & |h| \geq 2 
\end{cases} \]

(2) MA(2): \(X_t = Z_t + \theta_1 Z_{t-1} + \theta_2 Z_{t-2}\)

\[ \gamma_x(h) = E\{[X_t - E(X_t)][X_{t+h} - E(X_{t+h})]\} = E(X_t X_{t+h}) \]
\[ = E\{[Z_t + \theta_1 Z_{t-1} + \theta_2 Z_{t-2}][Z_{t+h} + \theta_1 Z_{t+h-1} + \theta_2 Z_{t+h-2}]\} = E(Z_t Z_{t+h}) + \theta_1 E(Z_t Z_{t+h-1}) + \theta_2 E(Z_t Z_{t+h-2}) + \theta_1 E(Z_{t-1} Z_{t+h}) + \theta_2 E(Z_{t-2} Z_{t+h}) + \theta_1 \theta_2 E(Z_{t-1} Z_{t+h-2}) + \theta_2 E(Z_{t-2} Z_{t+h-1}) + \theta_1 \theta_2 E(Z_{t-2} Z_{t+h-2}) \]
\[ = \begin{cases} 
(1 + \theta_1^2 + \theta_2^2)\sigma^2, & h = 0 \\
\theta_1(1 + \theta_2)\sigma^2, & |h| = 1 \\
\theta_2 \sigma^2, & |h| = 2 \\
0, & |h| \geq 3 
\end{cases} \]
\[ \rho_x(h) = \frac{\gamma_x(h)}{\gamma_x(0)} = \begin{cases} 
1, & h = 0 \\
\frac{\theta_1(1 + \theta_2)}{1 + \theta_1^2 + \theta_2^2}, & |h| = 1 \\
\frac{\theta_2}{1 + \theta_1^2 + \theta_2^2}, & |h| = 2 \\
0, & |h| \geq 3 
\end{cases} \]
(3) MA(3): \( X_t = Z_t + \theta_1 Z_{t-1} + \theta_2 Z_{t-2} + \theta_3 Z_{t-3} \)

\[
\gamma_x(h) = E\{[X_t - E(X_t)][X_{t+h} - E(X_{t+h})]\} = E(X_t X_{t+h})
\]

\[
= E\{[Z_t + \theta_1 Z_{t-1} + \theta_2 Z_{t-2} + \theta_3 Z_{t-3}][Z_{t+h} + \theta_1 Z_{t+h-1} + \theta_2 Z_{t+h-2} + \theta_3 Z_{t+h-3}]\}
\]

\[
= E(Z_t Z_{t+h}) + \theta_1 E(Z_t Z_{t+h-1}) + \theta_2 E(Z_t Z_{t+h-2}) + \theta_3 E(Z_t Z_{t+h-3})
\]

\[
+ \theta_1 E(Z_{t-1} Z_{t+h}) + \theta_1^2 E(Z_{t-1} Z_{t+h-1}) + \theta_1 \theta_2 E(Z_{t-1} Z_{t+h-2}) + \theta_1 \theta_3 E(Z_{t-1} Z_{t+h-3})
\]

\[
+ \theta_2 E(Z_{t-2} Z_{t+h}) + \theta_1 \theta_2 E(Z_{t-2} Z_{t+h-1}) + \theta_2^2 E(Z_{t-2} Z_{t+h-2}) + \theta_2 \theta_3 E(Z_{t-2} Z_{t+h-3})
\]

\[
+ \theta_3 E(Z_{t-3} Z_{t+h}) + \theta_1 \theta_3 E(Z_{t-3} Z_{t+h-1}) + \theta_2 \theta_3 E(Z_{t-3} Z_{t+h-2}) + \theta_3^2 E(Z_{t-3} Z_{t+h-3})
\]

\[
= \begin{cases} 
(1 + \theta_1^2 + \theta_2^2 + \theta_3^2)\sigma^2, & h = 0 \\
(\theta_1 + \theta_2 + \theta_3)\sigma^2, & |h| = 1 \\
(\theta_2 + \theta_1 \theta_3)\sigma^2, & |h| = 2 \\
\theta_3 \sigma^2, & |h| = 3 \\
0, & |h| \geq 4
\end{cases}
\]

\[
\rho_x(h) = \frac{\gamma_x(h)}{\gamma_x(0)} = \begin{cases} 
1, & h = 0 \\
\frac{\theta_1 + \theta_2 + \theta_3}{1 + \theta_1^2 + \theta_2^2 + \theta_3^2}, & |h| = 1 \\
\frac{\theta_2 + \theta_1 \theta_3}{1 + \theta_1^2 + \theta_2^2 + \theta_3^2}, & |h| = 2 \\
\frac{\theta_3}{1 + \theta_1^2 + \theta_2^2 + \theta_3^2}, & |h| = 3 \\
0, & |h| \geq 4
\end{cases}
\]

3.3 Show that the MA(1) processes \( X_t = Z_t + \theta Z_{t-1} \) and \( X_t = (Z_t + 1/\theta) Z_{t-1} \), with \( \theta \neq 0 \), have the same acf.

Hint: Use the general formulas for acf.

3.4 Find MA(q) representation for a stationary process with

a) acf \( \rho(h) = \begin{cases} 
1, & h = 0, \\
0.3, & |h| = 1, \\
0, & |h| > 1.
\end{cases} \)

Solution: We can use the general formula we found in (3.2) to determine the coefficients for an MA(1) stationary process. Computing the acf values from the general formula with the problem statement values
yields the following system of equations.

\[ \rho(h) = \begin{cases} 
1, & h = 0 \\
\frac{\theta}{1 + \theta^2}, & |h| = 1 \\
0, & |h| > 1 
\end{cases} \]

We can use the case where \(|h| = 1\) to solve for the only unknown value, \(\theta\).

\[ 0.3 = \frac{\theta}{1 + \theta^2} \quad \Rightarrow 0 = \theta^2 - \frac{1}{0.3} \theta + 1 \quad \Rightarrow \theta = \frac{1}{3} \]

Using this result, we can construct the following MA, which has the desired acf:

\[ X_t = Z_t + \frac{1}{3} Z_{t-1} \]

b) acvf \(\gamma(h) = \begin{cases} 
31/18, & h = 0 \\
-35/36, & |h| = 1 \\
1/6, & |h| = 2 \\
0, & |h| > 2 
\end{cases} \]

**Solution:** Similar to above, we can use the formula found in (3.2) to determine the coefficients for an MA(2) stationary process. Comparing the acf values from the general formula with the problem statement values yields the following system of equations.

\[ \gamma(h) = \begin{cases} 
(1 + \theta_1^2 + \theta_2^2)\sigma^2, & h = 0 \\
\theta_1(1 + \theta_2)\sigma^2, & |h| = 1 \\
\theta_2\sigma^2, & |h| = 2 \\
0, & |h| \geq 3 
\end{cases} \]

We can use the case where \(|h| = 2\) to solve for the unknown value, \(\theta_2\).

\[ \frac{1}{6} = \theta_2\sigma^2 \quad \Rightarrow \theta_2 = \frac{1}{6\sigma^2} \]

We can then use the case where \(|h| = 1\) to solve for the unknown value, \(\theta_1\).

\[ -\frac{35}{36} = \theta_1(1 + \theta_2)\sigma^2 = \theta_1\left(\frac{1}{6\sigma^2} + 1\right)\sigma^2 \quad \Rightarrow \theta_1 = \frac{-35}{6(1 + 6\sigma^2)} \]
Lastly, we can use the case where \( h = 0 \) to solve for the remaining unknown value, \( \sigma^2 \).

\[
\frac{31}{18} = (1 + \theta_1^2 + \theta_2^2) \\
= (1 + \frac{35^2}{36(1 + 6\sigma^2)^2} + \frac{1}{36\sigma^2}) \sigma^2 \\
= \frac{(36\sigma^4 + 1)(1 + 6\sigma^2)^2 + 35^2\sigma^4}{36(1 + 6\sigma^2)^2\sigma^2} \\
0 = (36\sigma^4 - 62\sigma^2 + 1)(1 + 12\sigma^2 + 36\sigma^4) + 35^2\sigma^4 \\
= 1296\sigma^8 - 1800\sigma^6 + 553\sigma^4 - 50\sigma^2 + 1 \\
\sigma^2 = 1, \frac{1}{4}, \frac{1}{9}, \frac{1}{36}
\]

Thus, if we let \( \sigma^2 = 1 \), (i.e. \( Z_t \sim WN(0,1) \)), we can construct the following MA, which has the desired acf:

\[
X_t = Z_t - \frac{35}{42}Z_{t-1} + \frac{1}{6}Z_{t-2}
\]

3.5 Find the acf of the following MA(2) processes:

a) \( X_t = Z_t + 0.7Z_{t-1} - 0.2Z_{t-2} \); Solution: From 4.2, we know the acf for an MA(2) process has the following form.

\[
\rho_x(h) = \begin{cases} 
1, & h = 0 \\
\theta_1(1+\theta_2), & |h| = 1 \\
\theta_2, & |h| = 2 \\
0, & |h| \geq 3
\end{cases}
\]

\[
\rho_x(h) = \begin{cases} 
1, & h = 0 \\
0.7(1-0.2), & |h| = 1 \\
0, & |h| = 2 \\
0, & |h| \geq 3
\end{cases}
\]

\[
\rho_x(h) = \begin{cases} 
1, & h = 0 \\
0.366, & |h| = 1 \\
-0.131, & |h| = 2 \\
0, & |h| \geq 3
\end{cases}
\]

b) \( X_t = Z_t + 0.3Z_{t-1} - 0.4Z_{t-2} \); Solution: From 4.2, we know the acf for an MA(2) process has the following form.

\[
\rho_x(h) = \begin{cases} 
1, & h = 0 \\
\theta_1(1+\theta_2), & |h| = 1 \\
\theta_2, & |h| = 2 \\
0, & |h| \geq 3
\end{cases}
\]

\[
\rho_x(h) = \begin{cases} 
1, & h = 0 \\
0.5(1-0.4), & |h| = 1 \\
0.144, & |h| = 2 \\
-0.32, & |h| = 2 \\
0, & |h| \geq 3
\end{cases}
\]

\[
\rho_x(h) = \begin{cases} 
1, & h = 0 \\
0.2, & |h| = 1 \\
-0.32, & |h| = 2 \\
0, & |h| \geq 3
\end{cases}
\]

c) \( X_t = Z_t - 0.4Z_{t-1} - 0.1Z_{t-2} \).
3.6 Consider the infinite-order MA process \( \{X_t\} \) defined by

\[
X_t = Z_t + C(Z_{t-1} + Z_{t-2} + \ldots),
\]

where \( C \) is a constant. Show that the process \( X_t \) is non-stationary. Show that the process \( Y_t = \nabla X_t \) is stationary and find its acf.

**Solution:** First let us consider the MA Process \( X_t \). The mean is given by

\[
E(X_t) = E\{Z_t + C(Z_{t-1} + Z_{t-2} + \ldots)\}
= E(Z_t) + C[E(Z_{t-1}) + E(Z_{t-2}) + \ldots]
= 0 + C[0 + 0 + \ldots] = 0
\]

and the autocovariance is equal to

\[
\gamma_x(h) = E\{[X_t - E(X_t)][X_{t+h} - E(X_{t+h})]\}
= E(X_t X_{t+h})
= E\{Z_t Z_{t+h} + C(Z_t Z_{t+h-1} + Z_t Z_{t+h-2} + \ldots) + C(Z_{t-1} Z_{t+h} + Z_{t-2} Z_{t+h} + \ldots) + C^2(Z_{t-1} Z_{t+h-1} + Z_{t-1} Z_{t+h-2} + \ldots) + C^2(Z_{t-2} Z_{t+h} + Z_{t-2} Z_{t+h-2} + \ldots) + \ldots\}
= \left\{ \begin{array}{ll}
(1 + \sum_{i=1}^{\infty} C^2)\sigma^2, & h = 0 \\
(C + \sum_{i=1}^{\infty} C^2)\sigma^2, & |h| > 0
\end{array} \right.
\]

Because \( \sum_{i=1}^{\infty} C^2 \) does not converge, the autocovariance for \( X_t \) is not finite, so the process fails the second necessary condition for weak stationarity. Let us now consider whether the process \( Y_t = \nabla X_t \) is stationary. First let us express \( Y_t \) in terms of the white noise process \( Z_t \).

\[
Y_t = X_t - X_{t-1}
= [Z_t + C(Z_{t-1} + Z_{t-2} + \ldots)] - [Z_{t-1} + C(Z_{t-2} + Z_{t-3} + \ldots)]
= Z_t + (C - 1)Z_{t-1}
\]

Note that \( Y_t \) is a MA(1) process. From the class notes we know that a finite MA(\( q \)) process is stationary for any set of real coefficients, which means that \( Y_t \) is stationary.
3.7 Find a stationary process with the autocorrelation function 
\( \rho(h) = (-1)^{|h|} \).

(A possible) Answer: \( X_t = (-1)^t Y \), where \( Y \sim N(0, \sigma^2) \).