Problem 1
The rvs $Y$ and $X$ are related as
\[ Y = 10 + 20X + \epsilon, \quad \epsilon \sim N(0,4^2). \]

a) Find the conditional distribution of $Y$ given $X = x$.
b) Find the conditional expectation of $Y$ given $X = 3$.
c) Find the conditional variance of $Y$ given $X = 0$.

Problem 2
Let $(X,Y)$ be a bivariate Normal random variable such that $Y \sim N(4,3^2)$, $X \sim N(5,2^2)$, and $\rho(X,Y) = 0.8$.

a) Find the conditional expectations $E(Y|Y = y)$, $E(X|X = x)$, $E(Y|X = x)$ and $E(X|Y = y)$.
b) Find the best mean-square constant forecast of $Y$, $\hat{Y} = c$.
c) Find the best mean-square forecast of $Y$ by a function of $X$, $\hat{Y} = f(X)$.
d) Find the best mean-square forecast of $X$ by a function of $Y$, $\hat{X} = f(Y)$.

Problem 3
Consider rvs $Y$ and $X$ with finite variances. We notice that when $\rho(X,Y) = \pm 1$, the rvs $Y$ and $X$ are (deterministically) linearly related, and the slopes of the lines $Y = \beta_0 + \beta_1 X$ and $X = \alpha_0 + \alpha_1 Y$ are reciprocal to each other:
\[ \alpha_1 \beta_1 = 1. \]

True or False: If $\rho(X,Y) \neq \pm 1$, the slopes of the best mean-square forecast line $\hat{Y} = \beta_0 + \beta_1 X$ and $\hat{X} = \alpha_0 + \alpha_1 Y$ are reciprocal to each other?

Problem 4
Consider rvs $Y$ and $X$ with finite variances.

a) Formulate the necessary and sufficient conditions for the best mean-square forecast of $Y$ by a linear function of $X$ to be $\hat{Y} = X$.
b) It is known that the best mean-square linear forecast of $Y$ by a (linear) function of $X$ is $\hat{Y} = X$, and $\text{Var}(Y) = 4$. Find the range of possible values for the standard deviation of $X$, $\sigma_X$.

Problem 5
Suppose that $X_t$ is a stationary time series with acf $\rho(h)$ and variance $\sigma^2$.

a) Show that the best mean-square forecast of $X_{t+h}$ in the form
\[ \hat{X}_{t+h} = aX_t + b \]
corresponds to $a = \rho(h)$, $b = E(X_0)(1 - \rho(h))$.
b) What is the mean-square error of this forecast?