1. 3.7.19b

\[ f_X(x) = \int_0^1 \frac{3}{2} y^2 \, dy = \frac{1}{2} \quad f_Y(y) = \int_0^2 \frac{3}{2} y^2 \, dx = 3y^2 \]

3.7.19d Integrating \( f_{X,Y}(x,y) \) over the unit interval for both \( x \) and \( y \) yields \( c = 1 \). Thus,

\[ f_X(x) = \int_0^1 (x + y) \, dy = x + \frac{1}{2} \quad f_Y(y) = y + \frac{1}{2} \quad \text{(by symmetry.)} \]

2. 3.7.23 Using the definition of the marginal distributions, then Wolfram Alpha to simplify,

\[
p_X(x) = \sum_{y=0}^{4-x} \frac{4!}{x!y!(4-x-y)!} \left( \frac{1}{2} \right)^x \left( \frac{1}{3} \right)^y \left( \frac{1}{6} \right)^{4-x-y} = \cdots = \frac{3}{2(4-x)!x!} = \left( \frac{4}{x} \right) \left( \frac{1}{2} \right)^x \left( \frac{1}{2} \right)^{4-x}
\]

and

\[
p_Y(y) = \sum_{x=0}^{4-y} \frac{4!}{x!y!(4-x-y)!} \left( \frac{1}{2} \right)^x \left( \frac{1}{3} \right)^y \left( \frac{1}{6} \right)^{4-x-y} = \cdots = \frac{2^{7-y}}{27(4-y)!y!} = \left( \frac{4}{y} \right) \left( \frac{1}{3} \right)^y \left( \frac{2}{3} \right)^{4-y}
\]

3. 3.7.26 Note that you’ll need to use the generalized hypergeometric PDF to compute the following probabilities (since there are 3 types, not just 2).

\[
F_{X,Y}(1,2) = P(X \leq 1, Y \leq 2) = \sum_{i=0}^{1} \sum_{j=0}^{2} P(\mathbf{X} = i, \mathbf{Y} = j) \\
= \left( \binom{5}{0} \binom{4}{1} \binom{3}{2} + \binom{5}{1} \binom{4}{2} \binom{3}{2} + \binom{5}{2} \binom{4}{3} \binom{3}{2} + \binom{5}{1} \binom{4}{1} \binom{3}{2} + \binom{5}{2} \binom{4}{2} \binom{3}{1} \right) \left( \binom{12}{4} \right) \\
= \frac{4 + 18 + 60 + 90}{495} = \frac{177}{495}
\]

4. 3.7.28(a)

\[
F_{X,Y}(u,v) = P(X \leq u, Y \leq v) = \int_0^u \int_x^v \frac{1}{2} \, dy \, dx = \int_0^u \left[ \frac{1}{2} \right]_x^v \, dy \\
= \int_0^u \frac{1}{2} (v - x) \, dx = \left[ \frac{1}{2} vx - \frac{x^2}{4} \right]_0^u = \frac{1}{2} vu - \frac{1}{4} u^2
\]
5. **BONUS 3.7.28(c)** This problem requires that you split into 2 cases: the situation where \((u,v)\) in the definition of the CDF falls in the range where the joint PDF is defined, and the case where \((u,v)\) is outside that range.

**Case 1:** \(v \leq 1 - u\)

\[
F_{X,Y}(u,v) = P(X \leq u, Y \leq v) = \int_0^u \int_0^v 6x\,dy\,dx
\]

\[
= \int_0^u \left[6xy\right]_0^v = \int_0^u 6xv\,dx = 3u^2v
\]

**Case 2:** \(v > 1 - u\)

\[
F_{X,Y}(u,v) = P(X \leq u, Y \leq v) = \int_0^u \int_0^v 6x\,dy\,dx - \int_0^u \int_{1-v}^{v} 6x\,dy\,dx
\]

\[
= 3u^2v - [3u^2v - 3u^2 + 2u^3 - 3(1-v)^2v + 3(1-v)^2 - 2(1-v)^3]
\]

\[
= 3u^2 - 2u^3 - (1-v)^3
\]

6. **3.7.38** To show independence, we must show that \(p_{X,Y}(j,k) = p_X(j)p_Y(k)\). For any pair \((j,k)\), we have that

\[
p_{X,Y}(j,k) = \frac{1}{36} = \left(\frac{1}{6}\right)\left(\frac{1}{6}\right) = p_X(j)p_Y(k).
\]

Hence, \(X\) and \(Y\) are independent random variables.

7. **3.7.43**

\[
P(Y < X) = \int_0^1 \int_0^x f_{X,Y}(x,y)\,dy\,dx = \int_0^1 \int_0^x (2x)(3y^2)\,dy\,dx = \int_0^1 2x^4\,dx = \frac{2}{5}
\]

8. **3.7.44** Since \(X\) and \(Y\) are independent,

\[
f_{X,Y}(x,y) = f_X(x) f_Y(y) = xy, \text{ for } x \in [0, 2] \text{ and } y \in [0, 1]
\]

\[
\Rightarrow F_{X,Y}(u,v) = \int_0^u \int_0^v xy\,dy\,dx = \int_0^u \left[x \cdot \frac{y^2}{2}\right]_0^v
\]

\[
= \int_0^u \frac{1}{2}v^2x\,dx = \frac{v^2}{2} \cdot \frac{x^2}{2} _0^u
\]

\[
= \frac{1}{4}u^2v^2 \text{ for } 0 \leq u \leq 2, 0 \leq v \leq 1
\]
9. **3.7.48** We need to show that the events $U = g(X) \in A$ and $V = h(Y) \in B$ are independent for sets of real numbers $A$ and $B$. Note that

$$P(g(X) \in A \text{ and } h(Y) \in B) = P(X \in g^{-1}(A) \text{ and } Y \in h^{-1}(B)).$$

Since $X$ and $Y$ are independent,

$$P(X \in g^{-1}(A) \text{ and } Y \in h^{-1}(B)) = P(X \in g^{-1}(A))P(Y \in h^{-1}(B)) = P(g(X) \in A)P(h(Y) \in B).$$

Thus, $U = g(X) \in A$ and $V = h(Y) \in B$ are independent.

10. **3.8.1(a)** The sum is also Poisson (with rate $\lambda + \mu$), since

$$p_{X+Y}(w) = \sum_{x=0}^{\infty} p_X(x)p_Y(w-x) = \sum_{x=0}^{w} p_X(x)p_Y(w-x)$$

$$= \sum_{x=0}^{w} e^{-\lambda} \frac{x^x}{x!} e^{-\mu} \frac{\mu^{w-x}}{(w-x)!} = e^{-(\lambda+\mu)} \frac{(\lambda + \mu)^w}{w!} \sum_{x=0}^{w} \frac{w!}{x!(w-x)!} \left( \frac{\lambda}{\mu + \lambda} \right)^x \left( \frac{\mu}{\mu + \lambda} \right)^{w-x}$$

$$= e^{-(\lambda+\mu)} \frac{(\lambda + \mu)^w}{w!} \cdot 1$$

$$= e^{-(\lambda+\mu)} \frac{(\lambda + \mu)^w}{w!}$$

Notice that the sum in line 2 of the equation above is the sum over all $x$ of a binomial PDF with parameters $n = w$ and $p = \frac{\lambda}{\mu + \lambda}$, which is 1 (property of a PDF).

11. **3.8.4** Since $V$ is independent of independent RVs $X$ and $Y$, then it is independent of their sum $W = X + Y$ since we can write the joint CDF as the product of the CDF for $V$ and the CDF for $W$. For the continuous case (and similarly for the discrete case), we have

$$F_{V,W}(v, w) = \int_{-\infty}^{v} \int_{-\infty}^{\infty} \int_{-\infty}^{w-y} f_V(v) f_X(x) f_Y(y) \, dx \, dy \, dv$$

$$= \left( \int_{-\infty}^{v} f_V(v) \, dv \right) \left( \int_{-\infty}^{\infty} \int_{-\infty}^{w-y} f_X(x) f_Y(y) \, dx \, dy \right)$$

$$= F_V(v) F_W(w)$$