Problem 1

Suppose that \( X \) and \( Y \) have a continuous joint distribution for which the joint PDF is as follow:

\[
f(x, y) = \begin{cases} 
\left( \frac{3}{2} \right) y^2 & \text{for } 0 \leq x \leq 2 \text{ and } 0 \leq y \leq 1, \\
0 & \text{OW}
\end{cases}
\]

(a)

\[
f_X(x) = \int_0^1 \frac{3}{2} y^2 \, dy = \left. \frac{1}{2} y^3 \right|_{y=0}^{y=1} = \frac{1}{2}, \quad 0 \leq x \leq 2
\]

\[
f_Y(y) = \int_0^2 \frac{3}{2} y^2 \, dx = \left. \frac{3}{2} y^2 x \right|_{x=0}^{x=2} = 3y^2, \quad 0 \leq y \leq 1
\]

(b) \( f_{X \mid Y=y}(x) = \frac{f(x, y)}{f_Y(y)} = \begin{cases} 
\frac{\frac{3}{2} y^2}{3y^2} = \frac{1}{2} & , \quad 0 \leq x \leq 2, \\
0 & , \text{OW.}
\end{cases} \)

\[
f_{Y \mid X=x}(y) = \frac{f(x, y)}{f_X(x)} = \begin{cases} 
\frac{\frac{3}{2} y^2}{3y^2} = 3y^2 & , \quad 0 \leq y \leq 1, \\
0 & , \text{OW.}
\end{cases}
\]

* Note that the conditional PDFs are not defined when \( f_Y(y) = 0 \) and \( f_X(x) = 0 \) respectively.

(c)

\[
P(X < 1 \mid Y = 0.5) = \int_0^1 f_{X \mid Y=0.5}(x) \, dx = \int_0^1 \frac{1}{2} \, dx = 1 \frac{1}{2} = \frac{1}{2}
\]

Problem 2

Suppose that \( X \) and \( Y \) have a continuous joint distribution for which the joint PDF is as follow:

\[
f(x, y) = \begin{cases} 
\left( \frac{15}{4} \right) x^2 & \text{for } 0 \leq y \leq 1 - x^2, \\
0 & \text{OW.}
\end{cases}
\]

(a)

\[
f_X(x) = \int_0^{1-x^2} \frac{15}{4} x^2 \, dy = \left. \frac{15}{4} x^2 y \right|_{y=0}^{y=1-x^2} = \frac{15}{4} x^2 (1 - x^2), \quad -1 \leq x \leq 1
\]

\[
f_Y(y) = \int_{-\sqrt{1-y}}^{\sqrt{1-y}} \frac{15}{4} x^2 \, dx = \left. \frac{5}{4} x^3 \right|_{x=-\sqrt{1-y}}^{x=\sqrt{1-y}} = 2.5(1 - y)^{\frac{3}{2}}, \quad 0 \leq y \leq 1
\]

(b) NO, \( X \) and \( Y \) are not independent, because \( f(x, y) \neq f_X(x)f_Y(y) \)
(c)\[ f_{X|Y=y}(x) = \frac{f(x,y)}{f_Y(y)} = \begin{cases} \frac{\frac{4}{5}x^2}{2.5(1-y)^2} = \frac{3x^2}{2(1-y)^2}, & -\sqrt{1-y} \leq x \leq \sqrt{1-y}, 0 \leq y < 1 \\ 0, & OW. \end{cases} \]
\[ f_{Y|X=x}(y) = \frac{f(x,y)}{f_X(x)} = \begin{cases} \frac{\frac{4}{5}x^2}{\frac{4}{5}x^2(1-x^2)} = \frac{1}{1-x^2}, & 0 \leq y \leq 1-x^2, 0 \leq x < 1 \\ 0, & OW. \end{cases} \]

* Note that the conditional PDFs are not defined when \( f_Y(y) = 0 \) and \( f_X(x) = 0 \) respectively.

**Problem 3**

There are 4 possibilities for choosing 2 marbles out of the 10 without replacement: (Red, Red), (Red, Green), (Green, Red), (Green, Green). We are given that \( X_i = 1 \) if the marble selected is red and 0 otherwise, so the above possibilities can be written as \( P(X_1 = i, X_2 = j) \) for \( i, j = 0, 1 \). Then we can compute each of the 4 probabilities and hence, the

(a) Joint PDF of \( X_1 \) and \( X_2 \) is
\[
\begin{align*}
\Pr(X_1, X_2 = 1, 1) &= \Pr(X_1 = 1, X_2 = 1) = \Pr(X_2 = 1|X_1 = 1) \cdot \Pr(X_1 = 1) = \frac{3}{9} \cdot \frac{4}{10} = \frac{12}{90} \\
\Pr(X_1, X_2 = 1, 0) &= \Pr(X_1 = 1, X_2 = 0) = \Pr(X_2 = 0|X_1 = 1) \cdot \Pr(X_1 = 1) = \frac{6}{9} \cdot \frac{4}{10} = \frac{24}{90} \\
\Pr(X_1, X_2 = 0, 1) &= \Pr(X_1 = 0, X_2 = 1) = \Pr(X_2 = 1|X_1 = 0) \cdot \Pr(X_1 = 0) = \frac{4}{9} \cdot \frac{6}{10} = \frac{24}{90} \\
\Pr(X_1, X_2 = 0, 0) &= \Pr(X_1 = 0, X_2 = 0) = \Pr(X_2 = 0|X_1 = 0) \cdot \Pr(X_1 = 0) = \frac{5}{9} \cdot \frac{6}{10} = \frac{30}{90}
\end{align*}
\]

(b) The marginal PDFs are
\[
\begin{align*}
\Pr(X_1 = 0) &= \frac{24}{90} + \frac{30}{90} = \frac{3}{5} \\
\Pr(X_1 = 1) &= \frac{12}{90} + \frac{24}{90} = \frac{2}{5} \\
\Pr(X_2 = 0) &= \frac{24}{90} + \frac{30}{90} = \frac{3}{5} \\
\Pr(X_2 = 1) &= \frac{12}{90} + \frac{24}{90} = \frac{2}{5}
\end{align*}
\]

(c) \( X_1 \) and \( X_2 \) are not independent since, for example, \( \Pr(X_1, X_2 = 0, 0) \neq \Pr(X_1 = 0) \Pr(X_2 = 0) \).

(d) The conditional PDFs are
\[
\begin{align*}
\Pr(X_1|X_2 = 1) &= \frac{\Pr(X_1, X_2 = 1, 1)}{\Pr(X_2 = 1)} = \frac{2}{3} \\
\Pr(X_1|X_2 = 1) &= \frac{\Pr(X_1, X_2 = 1, 1)}{\Pr(X_2 = 1)} = \frac{1}{3} \\
\Pr(X_2|X_1 = 0) &= \frac{\Pr(X_1, X_2 = 0, 0)}{\Pr(X_1 = 0)} = \frac{5}{9} \\
\Pr(X_2|X_1 = 0) &= \frac{\Pr(X_1, X_2 = 0, 1)}{\Pr(X_1 = 0)} = \frac{4}{9}
\end{align*}
\]
Problem 4

\[ P(Y^{(2)} > y_{60}) = 1 - P(Y^{(2)} \leq y_{60}) = 1 - p(Y_1 \leq y_{60}, Y_2 \leq y_{60}) \]

\[ = 1 - P(Y_1 \leq y_{60})P(Y_2 \leq y_{60}) \text{ since } Y_1 \text{ and } Y_2 \text{ are independent} \]

\[ = 1 - (0.6)(0.6) = 0.64 \]

Problems 5 and 6

Recall that \( X_i \) for \( i = 1, 2, \ldots, n \) are \( i.i.d \) and \( Y = X_1 + \ldots + X_n \).

(a) Let \( X_i \sim \text{Poisson}(x; \lambda) \) such that \( f(x) = \frac{e^{-\lambda x} \lambda^x}{x!}, x = 0, 1, 2, \ldots \)

Then,

\[ f(x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} \frac{e^{-\lambda x_i} \lambda^{x_i}}{x_i!} = \frac{e^{-n\lambda \sum x_i} \lambda^{\sum x_i}}{\prod_i x_i!} \]

Since, \( M(t) = e^{\lambda(e^t - 1)} \) then,

\[ M_Y(t) = \prod_i e^{\lambda(e^t - 1)} = e^{n\lambda(e^t - 1)} \]

Hence, \( Y \sim \text{Poisson}(y; n\lambda) \).

(b) Let \( X_i \sim \text{Bern}(x; p) \) such that \( f(x) = p^x(1-p)^{1-x}, x = 0, 1 \)

Then,

\[ f(x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} p^x(1-p)^{1-x} = p^{\sum x_i} (1-p)^{\sum (1-x_i)} \]

Since, \( M(t) = (1-p + pe^t) \) then,

\[ M_Y(t) = \prod_i (1-p + pe^t) = (1-p + pe^t)^n \]

Hence, \( Y \sim \text{Bin}(y; n,p) \).

(c) Let \( X_i \sim \text{Geo}(x; p) \) such that \( f(x) = p(1-p)^{x-1}, x = 1, 2, \ldots \)

Then,

\[ f(x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} p(1-p)^{x_i-1} = p^{\sum x_i} (1-p)^{\sum x_i - n} \]

Since, \( M(t) = \left( \frac{pe^t}{1-(1-p)e^t} \right) \) then,

\[ M_Y(t) = \prod_i \left( \frac{pe^t}{1-(1-p)e^t} \right) = \frac{p^n e^{nt}}{(1-(1-p)e^t)^n} \]

Hence, \( Y \sim \text{NB}(y; n,p) \).

(d) Let \( X_i \sim G(x; \alpha, \beta) \) such that \( f(x) = \frac{1}{\beta \Gamma(\alpha)} x^{\alpha-1} e^{-\frac{x}{\beta}}, x > 0 \)

Then,

\[ f(x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} \frac{x_i^{\alpha-1} e^{-\frac{x_i}{\beta}}}{\beta \Gamma(\alpha)}, x_i > 0 \]
Since, $M(t) = \left(\frac{\beta}{\beta-t}\right)^\alpha$ then,

$$M_Y(t) = \prod_{i}^{n} \left(\frac{\beta}{\beta-t}\right)^{n\alpha}$$

Hence, $Y \sim G(y; n\alpha, \beta)$. 