Random Signals

• Some physical signals (noise) cannot be expressed as an explicit mathematical formula.
• These signals must be described in probabilistic terms.
• Noise is typically unwanted and to eliminate it we need to understand it quantitatively: we need probability theory.

Classical (Intuitive) Definition of Probability

• Assume all outcomes are equally likely
• \( N_A = \) number of ways event \( A \) can occur
• \( N = \) total number of possible outcomes

\[
P(A) = \frac{N_A}{N} \in [0,1]
\]

Relative Frequency Definition of Probability

• Perform an experiment \( N \) times (\( N \) large)
• Count the number of times \( N_A \) that \( A \) occurs

\[
P(A) = \frac{N_A}{N} \in [0,1]
\]
Problems: Relative Frequency

• Experiments cannot be performed an infinite (very large) number of times
• Assumption that the relative frequency approaches a constant limit (in practice the ratio hovers around a constant value)

Axiomatic Definition of Probability

• Accept a set of axioms based on experience
• Derive a complete theory based on axioms
• Random Experiment: experiment with nondeterministic outputs.
• Sample Space: set of outcomes of random experiment $s_i \in S$ (elementary events $s_i$) finite, countably infinite, or infinite

Sample Space Examples

Experiment. Single draw from 52-card deck
52 possible outcomes $\Rightarrow$ 52 elements in $S$

Experiment. (Vector Outcome) Throw of two fair dice observe number on each die,
$6 \times 6 = 36$ possible outcomes
$\Rightarrow 36$ elements in $S$, each a 2-tuple
$(1,1), \ldots, (1,6), (2,1), \ldots, (2,6), \ldots, (6,6)$

Event

• Assume elements of sample space are disjoint.
• Event $E = \text{subset of sample space (subject to constraints)}$ e.g. $E = \{S_i, i = 1,3,4\}$
• $S = \{S_i, i = 1,2,3,4,\ldots, 15\}$
• Events need not be disjoint
• Event Space : set of subsets of $S$ called events such that union, intersection, complement of an event is an event (called sigma field or sigma algebra if infinite union is not included).
Probability Space

- **Discrete Problem:** finite or countably infinite outcomes (complications for continuous case with uncountably infinite outcomes).
- **Probability Space:**
  1. Sample space
  2. Event space
  3. Probability measure

Axioms of Probability

\[
\text{i) } P(A) \geq 0 \\
\text{ii) } P(S) = 1 \\
\text{iii) } P\left(\bigcup_{i} A_i\right) = \sum_{i} P(A_i)
\]

\{A_i\} disjoint
\{A_i\} finite or countably infinite

Set Operations in Probability Theory

- Union: \(A \cup B\)
- Intersection: \(A \cap B\)
- Complement: \(A^c\)

\[
P(A \cup B) = P(A) + P(B) - P(A \cap B)
\]

Example: Throw of Die

- Sample Space
  \(S = \{1,2,3,4,5,6\}\) 6 elements
- Event Set = \(\{S, \emptyset, \{1\}, \ldots, \{6\}, \{1,2\}, \ldots, \{1,6\}, \ldots\}\) = Power Set of \(S\)
- \(A_1 = 1 \cup 5 \Rightarrow P(A_1) = P(1) + P(5) = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}\)
Marginal Probability

Written in margin outside \( m \) by \( n \) array for
\[
P(A_i), \ i = 1,2,\ldots,m \quad P(B_j), \ j = 1,2,\ldots,n
\]
\[
P(A_i) = \sum_{j=1}^{n} P(A_i \cap B_j), \quad i = 1,2,\ldots,m
\]
\[
P(B_j) = \sum_{i=1}^{m} P(A_i \cap B_j), \quad j = 1,2,\ldots,n
\]
\[
\sum_{i=1}^{m} P(A_i) = \sum_{j=1}^{n} P(B_j) = 1
\]

Conditional Probability

- Probability of event \( A \) given that event \( B \) has occurred (i.e. \( B \) = certain event, not defined for \( B \) impossible).
- Definition agrees with relative frequency notion.

\[
P(A / B) = \frac{P(A \cap B)}{P(B)}
\]

\[
P(A / B) = \frac{N_{\text{outcomes with both } A \text{ and } B}}{N_{\text{total}}}
\]

Bayes’ Theorem

\( A_i, \ i = 1,2,\ldots,m \), mutually exclusive events

\[
P(A_i / B) = \frac{P(B / A_i)P(A_i)}{\sum_{j=1}^{m} P(B / A_j)P(A_j)}
\]

\[
i) \ P(A_i \cap B) = P(B / A_i)P(A_i)
\]

\[
ii) P(B) = \sum_{j=1}^{m} P(B \cap A_j) = \sum_{j=1}^{m} P(B / A_j)P(A_j)
\]

Independence

Occurrence of one event does not affect the likelihood of the other

\[
P(A / B) = P(A) = \frac{P(A \cap B)}{P(B)}
\]

\[
P(A \cap B) = P(A)P(B)
\]

\[
P(B / A) = P(B)
\]
Multiple Random Variables

**Multivariate**: vector of random variables

**Bivariate**: 2 variables

Discrete case: joint prob. = 2-dim. array

Obtain marginal prob. by adding col. or row

\[ p_{X,Y}(x_i, y_j) = P(X = x_i, Y = y_j) \]

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Probability Distribution Function

**Definition**: The probability distribution function (PDF) of a random variable \( X \) is a function defined for each real number \( x \) as follows

\[ F(x) = P(X \leq x), -\infty < x < \infty \]

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Properties of the PDF

1. \( F(x) \to 0 \) as \( x \to -\infty \)
2. \( F(x) \to 1 \) as \( x \to \infty \)
3. \( F(x) \) is a nondecreasing function of \( x \).

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Probability Density Function

For a continuous random variable \( X \), there exists a nonnegative function \( f \) defined on the real line such that for every subset \( A=(a,b] \) of the real line the probability that \( X \) takes a value in \( A \). The function \( f \) is called the probability density function (pdf)

\[ P(a < X \leq b) = \int_a^b f_X(x)dx \]
Properties of the pdf

1. $\int_{-\infty}^{\infty} f_X(x) \, dx = 1$
2. $f_X(x) \geq 0$, $\forall x$.
3. $f_X(x) = \frac{dF_X(x)}{dx}$

Uniform Distribution

$f_X(x) = \begin{cases} \frac{1}{b-a} & a \leq x \leq b \\ 0, & \text{elsewhere} \end{cases}$

- Probability of value in any subinterval of $[a,b]$ is proportional to its length.
- Area of rectangle must be unity.

Expectation of a Random Variable

Expected value or mean of $X$.

$E(X) = \sum_i x_i P(x_i)$

$E(X) = \int_{-\infty}^{\infty} x \, f_X(x) \, dx$

Function of a Random Variable

Obtain the expected value of the function using

$E(X) = \sum_i g(x_i) P(x_i)$

$E(X) = \int_{-\infty}^{\infty} g(x) f_X(x) \, dx$
### Moments

- $k^{th}$ moment
- First moment is the mean

\[
E(X^k) = \sum_i x_i^k P(x_i)
\]

\[
E(X^k) = \int_{-\infty}^{\infty} x^k f_X(x) dx
\]

### Variance

Second moment about the mean

\[
Var(X) = E[(X - E[X])^2]
\]

\[
Var(X) = \sum_i (x_i - E[X])^2 P(x_i)
\]

\[
Var(X) = \int_{-\infty}^{\infty} (x - E[X])^2 f_X(x) dx
\]

### Properties of the Variance

\[
Var(aX + b) = a^2 Var(X)
\]

\[
Var(X) = E\{X^2\} - E^2\{X\}
\]

\[
Var(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} Var(X_i), \text{ } X_i \text{ independent}
\]

### Normal or Gaussian Density

\[
f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x - m)^2}{2\sigma^2}\right)
\]

\[
\Phi(x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt
\]

- Important pdf because:
  - Fits many physical phenomena.
  - Central limit theorem.
  - Completely described by mean and variance.
Right Tail Probability

- Probability of exceeding a given value.
- Complementary cumulative distribution.

\[ Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-t^2/2} dt \]

\[ = 1 - \Phi(x) \]

Inverse

- \( Q \) is monotonically decreasing and hence invertible.
- Inverse is important in signal detection.

\[ Q(\gamma) = \frac{1}{\sqrt{2\pi}} \int_0^\gamma e^{-t^2/2} dt = P_{FA} \]

\[ \gamma = Q^{-1}(P_{FA}) \]

Error Function

\( \text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt \)

\( \text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt = 1 - \text{erf}(x) \)

\( \Phi(x) = 0.5 \left[ 1 + \text{erf} \left( \frac{x}{\sqrt{2}} \right) \right] \)

\( Q(x) = 0.5 \text{erfc} \left( \frac{x}{\sqrt{2}} \right) \)

\( Q^{-1}(P) = \sqrt{2} \text{erfinv}(1 - 2P) \)

MATLAB

(similar for Maple)

\texttt{>> erf(x)} % Error function
\texttt{>> erfc(x)} % Complementary error function
\texttt{>> 0.5*(1+erf(x/sqrt(2)))} % St. Normal P (t < x)
\texttt{>> 0.5*erfc(x/sqrt(2))} % St. Normal P (t > x)
\texttt{>> sqrt(2)*erfinv(1-2*P)} % Inverse Q(P)
Chi-Square Distribution (Central)

\[ y = \sum_{i=1}^{n} x_i^2, x_i \sim N(0,1) \]

\[ f_{y}(y) = \begin{cases} \frac{1}{\Gamma(n/2)} y^{(n/2)-1} e^{-y/2}, & y > 0 \\ 0, & y < 0 \\ \end{cases} \]

\[ \Gamma(u) = \int_{0}^{\infty} t^{u-1} e^{-t} dt = (u-1)!, u \text{ integer} \]

\( n \) degrees of freedom

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Chi-Square (Noncentral)

\[ y = \sum_{i=1}^{n} x_i^2, x_i \sim N(m_i,1) \]

\[ \lambda = \sum_{i=1}^{n} m_i^2 \]

\[ f_{y}(y) = \begin{cases} \frac{1}{\Gamma(n/2)} \exp\left(-\frac{x+\lambda}{2}\right) I_{\lambda}(\sqrt{y}) x > 0 \\ 0, & y < 0 \end{cases} \]

\[ I_{\lambda}(u) = \frac{\lambda^{\lambda/2} y^{(\lambda/2)-1}}{\pi^{\lambda/2} \Gamma(\lambda/2)} \int_{0}^{\pi} \exp[\lambda \cos(t)] \sin^{\lambda-1}(t) dt \]

\( n \) degrees of freedom

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Rician Distribution

\[ x = \sqrt{x_1^2 + x_2^2}, \quad x_i \sim N(m_i, \sigma^2), i = 1,2 \]

\[ f_{\alpha}(x) = \begin{cases} \frac{x}{\sigma^2} \exp\left(-\frac{x^2 + \alpha^2}{2 \sigma^2}\right) \int_{0}^{\alpha \sigma} f_{\alpha}(x) dx, & x > 0 \\ 0, & x < 0 \end{cases} \]

\[ \alpha^2 = m_1^2 + m_2^2 \]

\[ I_{\alpha}(u) = \frac{1}{\pi} \int_{0}^{\pi} \exp(u \cos(\theta)) d\theta \]

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Multivariate Distributions

\[ X = n \text{ by } 1 \text{ vector} \]

\[ F(x) = P(X \leq x) \]

\(-\infty < x_i < \infty, i = 1, \cdots, n \)

\[ P(x \in A) = \int_{A} \cdots \int f(x) dx \]
Marginal Distributions

Marginal pdf

\[ f_i(x_i) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x) \, dx_{i+1} \cdots dx_n \]

Conditional Distribution

- For \( X_i, i = 1, \ldots, n \), conditional density given \( X_i = x_i, i = 2, \ldots, n \)

\[ f_{i\mid i-1}(x_i \mid x_{i-1}, \ldots, x_1) = \frac{f_{i-1}(x_i, x_{i-1}, \ldots, x_1)}{f_{i-1}(x_{i-1}, \ldots, x_1)} \]

Correlation and Covariance

- Generalization of second moment and variance to vector case.

\[
R_X = E\{x^T x\} \\
C_X = E\{(x - \mu_s)(x - \mu_s)^T\} \\
C_X = R_X, \mu_s = 0
\]

Multivariate Normal

- Generalization of normal distribution to \( n \) linearly independent random variables.

\[
f_X(x) = \frac{1}{(2\pi)^{n/2} |\Sigma|^{1/2}} \exp\left\{ -\frac{1}{2} (x - \mu)^T \Sigma^{-1} (x - \mu) \right\} \\
x = [X_1 \cdots X_n]
\]
Transformation of Random Variables

\[ \mathbf{y} = \mathbf{g}(\mathbf{x}) \]

\[ \mathbf{y} = [y_1 \ldots y_n]^T \]

\[ \mathbf{x} = [x_1 \ldots x_n]^T \]

- What is the pdf of the vector \( \mathbf{y} \)?
- How can we obtain it from the pdf of \( \mathbf{x} \)?

\[
\begin{align*}
\mathbb{P}(\mathbf{y} \in \mathcal{A}) &= \int_{\mathbf{y}^{-1}(\mathcal{A})} f_{\mathbf{x}}(\mathbf{x}) \left| \frac{\partial \mathbf{y}}{\partial \mathbf{x}} \right| d\mathbf{x} \\
&= \int_{\mathbf{y}^{-1}(\mathcal{A})} f_{\mathbf{x}}(\mathbf{x}) \left| \frac{\partial \mathbf{y}}{\partial \mathbf{x}} \right| d\mathbf{x} \\
&= \sum_{i=1}^{n} f_{i}(x_i) \left| \frac{\partial \mathbf{y}}{\partial \mathbf{x}} \right| \\
&= \sum_{i=1}^{n} f_{i}(x_i) \left| \frac{\partial \mathbf{y}}{\partial \mathbf{x}} \right|
\end{align*}
\]

Procedure

- Solve for \( \mathbf{x} \)
  a) If no solution exists, then \( f_{i} = 0 \)
  b) If one or more solutions exist

\[
\begin{align*}
\mathbf{y} = \mathbf{g}(\mathbf{x}) &\Rightarrow \mathbf{x}_{i}(\mathbf{y}), i = 1 \ldots n \\
f_{i}(\mathbf{y}) &= \sum_{i=1}^{n} f_{i}(\mathbf{x}) \left| \frac{\partial \mathbf{y}}{\partial \mathbf{x}} \right| \\
&= \sum_{i=1}^{n} f_{i}(\mathbf{x}) \left| \frac{\partial \mathbf{y}}{\partial \mathbf{x}} \right|
\end{align*}
\]

Transformation of Normal RV

- Normality is lost if the transformation is nonlinear.
- Normality is preserved if the transformation is linear.

\[
f_{\mathbf{x}}(\mathbf{x}) = \frac{1}{(2\pi)^{n/2} |\mathbf{C}|^{1/2}} \exp \left\{ -\frac{1}{2}(\mathbf{x} - \mathbf{m})^T \mathbf{C}^{-1}(\mathbf{x} - \mathbf{m}) \right\}
\]

\[ \mathbf{y} = \mathbf{A} \mathbf{x} + \mathbf{b} \Rightarrow \mathbf{x} = \mathbf{A}^{-1}(\mathbf{y} - \mathbf{b}) \]

\[ \mathbf{y} \sim \mathcal{N}(\mathbf{A}\mathbf{m} + \mathbf{b}, \mathbf{A}\mathbf{C}\mathbf{A}^T) \]

References

- Brown & Hwang
- Stark & Woods
- Gray & Davisson